Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs

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January 26, 2012

Abstract

In this paper, we study the stability and convergence of some general quadratic semimartingales. Motivated by financial applications, we study simultaneously the semimartingale and its opposite. Their characterization and integrability properties are obtained through some useful exponential submartingale inequalities. Then, a general stability result, including the strong convergence of the martingale parts in various spaces ranging from $\mathbb{H}^1$ to BMO, is derived under some mild integrability condition on the exponential of the terminal value of the semimartingale. This can be applied in particular to BSDE-like semimartingales.

This strong convergence result is then used to prove the existence of solutions of general quadratic BSDEs under minimal exponential

*Author partly supported by the "Chaire Financial Risk" of the Risk Foundation, Paris.

integrability assumptions, relying on a regularization in both linear-quadratic growth of the quadratic coefficient itself. On the contrary to most of the existing literature, it does not involve the seminal result of Kobylanski [30] on bounded solutions.

1 Introduction

The Backward Stochastic Differential Equations (BSDEs) were first introduced by Peng & Pardoux [42] in 1990 in the Lipschitz continuous framework, and then extended to continuous with linear growth framework by Lepeltier & San Martin [32] in 1997. They have been soon recognized as powerful tools with many different possible applications. More recently, there has been an accrued interest for quadratic BSDEs, with various fields of application such as risk sensitive control problems or dynamic financial risk measures and indifference pricing in mathematical finance.
In this case, the BSDE is an equation of the following type:

$$-\, dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T, \tag{1}$$

where $W_\cdot$ is a standard Brownian motion, and the coefficient g satisfies the following quadratic *structure condition* $\mathcal{Q}(l_\cdot, c_\cdot, \delta)$:

$$|\, g(t, y, z)| \leqslant \kappa(t, y, z) \equiv \frac{1}{\delta} l_t + c_t|y| + \frac{\delta}{2}|z|^2 \quad d\mathbb{P} \otimes dt\text{−a.s.}, \tag{2}$$

where $\delta > 0$ is a given constant, and $(l_t), (c_t)$ are predictable non-negative processes.
The first result concerning the existence and uniqueness of solutions to these equations was obtained in the bounded case in a Brownian filtration setting by Kobylanski [30] in 2000. The proof first relies on an exponential transformation as to come back to the better known framework of BSDEs with a coefficient with linear growth and then uses a regularization procedure to take the limit. The major difficulty is then about proving the strong convergence of the martingale parts without having to impose too strong assumptions. This seminal paper has been extended in several directions, to a continuous setting by Morlais [39], to unbounded solutions by Briand & Hu [7] or more recently by Mochel & Westray [38]. Some other authors have obtained further results in some particular situations (see for instance Hu & Schweizer [28], Hu, Imkeller & Muller[27], Mania and Tevzadze [36] or Delbaen, Hu &

Richou [13]). Recently in 2008, Tevzadze [45] has given a direct proof for the existence and uniqueness of a bounded solution in the Lipschitz-quadratic case.

We adopt in this paper a completely different approach and consider a forward point of view to treat directly the questions of convergence. To do so, we introduce the notion of general quadratic semimartingales in Section 2 and study their characterization with regards to their integrability properties under some interesting exponential transformations in Section 3. Mainly motivated by financial applications, where a seller price and a buyer price have to be given simultaneously, we apply systematically the same assumptions on the semimartingale and on its opposite. Having both exponential integrability properties proves to be essential in the a priori estimation of their quadratic variations. In Section 4, we obtain a general stability result, including the strong convergence of the martingale parts as presented in Theorem 4.5. The result is very general and simply require the existence of exponential moment of the absolute value of (or quantities related to) the terminal value of the semimartingales. Our approach allows us to obtain the strong convergence of the martingale parts in $\mathbb{H}^1$. Stability results are also obtained in various spaces, depending on the assumption made on the terminal values. It is interesting to note that, on the contrary to most of the existing literature, the space of BMO martingales does not play any particular role as the semimartingales are no longer bounded. This stability result is completed, in the BSDE framework, by the convergence in total variation of the finite variation part. In Section 5, existence results become a possible application of this stability result. More precisely, coming back to our initial motivation of quadratic BSDEs, we first regularize the quadratic coefficient of the BSDE through inf-convolution as to transform it into a coefficient with linear-quadratic growth. This regularization as linear-quadratic, and not simply linear, allows us to consider situations which are typically not considered in the literature. Applying the stability result of the previous section, we can pass to the limit and prove the existence result for general quadratic BSDEs, under "minimal" integrability assumptions. The power of the forward point of view is striking as existence results are easily obtained in a more general framework than the classical existing literature. However, uniqueness results requires stronger assumptions on the solutions, as in Kobylanski [30] for the bounded case, or for convex BSDEs, as in Briand & Hu [7] or more recently in Mochel & Westray [38] with exponential moments

of any order, or in Delbaen, Hu, & Richou [13] under weaker integrability assumptions.
This approach has also other potential applications that we will not discuss here for lack of space. We can just mention numerical simulations of quadratic BSDEs, study in terms of risk measures and dual representation, solving of associated HJB-type equations...

2 Quadratic semimartingales

Quadratic BSDEs have recently received a lot of attention, mainly due to the wide range of possible applications, involving optimization problems with an exponential criterion, such as risk-sensitive control problems introduced by Fleming in the 1980s (see Fleming & Sheu [21] for financial applications, or El Karoui & Hamadène for an application to risk-sensitive zero-sum stochastic functional games [17]).
Financial applications have generated a renewed interest for this type of BSDEs, particularly in connection with the theory of dynamic risk measures as in Barrieu & El Karoui [6], or indifference pricing with exponential utility (see for instance Rouge & El Karoui [44], Mania & Schweizer [35] or the recent book edited by Carmona [9] among many other references). Therefore, it is particularly relevant to understand the structure of these processes, and to obtain conditions ensuring their stability.
In the classical martingale theory, Burkolder-Davis-Gundy-type estimates are crucial to obtain convergence results for martingales in $\mathbb{H}^p$ from the convergence of their terminal values. The study of classical BSDEs with linear growth relies also on precise a priori estimates coming from the martingale theory, arising from a forward point of view (see for instance, in a general framework, El Karoui & Huang [19]). In this section, after having defined quadratic BSDEs, we adopt a forward point of view, introducing quadratic semimartingales, with a similar structure condition, studying their main properties and deriving some characterization results, which depend on various integrability assumptions. These results will be very useful to derive some stability and convergence results in the next section.

2.1 Definition of quadratic BSDEs and quadratic semimartingales

Let us briefly recall the definition of a quadratic BSDE. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ be a filtered probability space, where the filtration $(\mathcal{F}_t)$ satisfies the usual conditions of completeness and right-continuity. The σ-field on $\Omega \times \mathbb{R}^+$ generated by the adapted and left continuous processes is called the predictable σ-field and denoted by $\mathcal{P}$. In all the paper, we only consider *continuous* filtered probability space i.e. a filtered probability space such that any locally bounded martingale is a continuous martingale. A classical example is the probability space generated by a Brownian motion, and satisfying the usual conditions.

Definition of quadratic BSDEs A quadratic BSDE is an equation of the following type:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T, \tag{3}$$

where $T > 0$ is a given (possibly random) future time, $W_.$ is a standard d-dimensional $(\mathbb{P}, (\mathcal{F}_t))$-Brownian motion, and $Z_t dW_t$ simply denotes the scalar product. The random variable $\xi_T \in \mathcal{F}_T$ is the terminal condition, and the coefficient g is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^d)$ measurable process satisfying the following quadratic *structure condition* $\mathcal{Q}(l, c, \delta)$:

$$|\, g(., t, y, z)| \leqslant \kappa(t, y, z) \equiv |l_t| + c_t|y| + \frac{\delta}{2}|z|^2 \quad d\mathbb{P} \otimes dt\text{−a.s.}, \tag{4}$$

where $\delta > 0$ is a given constant, and $(l_.), (c_.)$ are predictable positive [1] processes.

By solution to the BSDE(g, ξ_T) defined in Equation (3), we mean a pair of predictable processes taking values in $\mathbb{R} \times \mathbb{R}^d$, $(Y, Z) = \{(Y_t, Z_t); t \in [0, T]\}$, such that the paths of Y are continuous, $\int_0^T |Z_t|^2 dt < \infty$, $\int_0^T |g(t, Y_t, Z_t)| dt < \infty$ hold $\mathbb{P}$-a.s., and

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{5}$$

This minimal definition will be completed later on by some further integrability assumptions.

[1] In the rest of the paper, we adopt the following European terminology: a positive random variable X verifies $\mathbb{P}(X \geqslant 0) = 1$, and a strictly positive random variable verifies $\mathbb{P}(X > 0) = 1$.

Definition of quadratic semimartingales Adopting a forward point of view, a solution of a quadratic BSDE is a quadratic Itô's semimartingale $Y_.$, where the predictable process with finite variation satisfies the same quadratic structure condition (4). Such a condition needs to be further specified when considering the more general framework of quadratic semimartingales defined on a continuous filtered probability space.

Definition 2.1 (Quadratic semimartingale). *Let $Y_.$ be a continuous semimartingale, with the decomposition $Y_. = Y_0 - V_. + M_.$, where $V_.$ is a predictable process with finite total variation $|V|_.$ and $M_.$ is a local martingale with quadratic variation $\langle M\rangle_.$.*
$Y_.$ is a quadratic semimartingale *if there exist two adapted continuous increasing processes $\Lambda.$ and $C.$ and a positive constant δ, such that the structure condition $\mathcal{Q}(\Lambda, C, \delta)$ holds true:*

$$d|V|_t \ll \frac{1}{\delta} d\Lambda_t + |Y_t|\, dC_t + \frac{\delta}{2} d\langle M\rangle_t, \quad d\mathbb{P}-\text{a.s.}. \tag{6}$$

The symbol $\ll$ stands for the strong order of increasing processes, stating that the difference is an increasing process. Sometimes we use the short notation $D_.^{\Lambda,C}(Y,\delta) = \frac{1}{\delta}\Lambda_. + |Y_.| * C_.$. At this stage, no particular integrability assumption is made on the processes $\Lambda_.$ and $C_.$.

Comments: (i) Observe that if $Y_.$ is a quadratic semimartingale, then $-Y_.$ is also a quadratic semimartingale
(ii) More generally, if $Y_.$ is a quadratic semimartingale and $\delta > 0$, $\delta Y_.$ is a semimartingale associated with $M_.^\delta = \delta M_.$ with quadratic variation $\langle M^\delta\rangle_. = \delta^2 \langle M\rangle_.$ and $V_.^\delta = \delta V$. Then the structure condition for the process $\delta Y_.$ becomes $d|V^\delta|_t \ll d\Lambda_t + |Y_t^\delta|\, dC_t + \frac{1}{2} d\langle M^\delta\rangle_t$. This property justifies our choice of restricting our study to quadratic semimartingales with constant $\delta = 1$, without any loss of generality.
(iii) The following notations specifiy different classes of quadratic semimartingales, $\mathcal{Q}(\Lambda, C, \delta)$ for the general case, $\mathcal{Q}(\Lambda, C)$ when $\delta = 1$, $\mathcal{Q}$ when $\Lambda_. \equiv 0, C_. \equiv 0, \delta = 1$.

2.2 Exponential transformations and algebraic characterization of quadra-tic semimartingales

Some recalls on semimartingales on a continuous probability space
(i) Let us first recall the conventional notation for the exponential martingale

of a continuous (local) martingale $M_.$ with quadratic variation $\langle M\rangle_.$

$$\mathcal{E}(M)_. \equiv \exp(M_. - \frac{1}{2}\langle M\rangle_.) \tag{7}$$

(ii) A right continuous left limited submartingale (càdlàg in the French denomination) $S_.$ is a càdlàg optional process $S_. = S_0 + N_. + K_.$, where $N_.$ is a local martingale and $K_.$ a predictable càdlàg increasing process. The pair $(N_., K_.)$ is called the additive decomposition of S. When $S_.$ is a positive submartingale, $(M_., A_.)$ is said to be the multiplicative decomposition of $S_.$ if $S_. = S_0\,\mathcal{E}_.(M)\exp(A_.)$, where $M_.$ is a local martingale and $A_.$ a predictable càdlàg increasing process.

(iii) Dellacherie & Meyer [15] (in Appendix 1 - Probabilités et Potentiel B) have extended this definition to right and left limited submartingales (also known as strong submartingales) when the increasing predictable process $K_.$ is only with left and right limits (làdlàg in the French denomination), with the following decomposition $K_. = K^1_. + K^2_{-.}$, where $K^1_.$ is a càdlàg predictable increasing process and $K^2_{-.}$ is the process of the left limits of a càdlàg optional increasing process $K^2_.$.

Characterization of $\mathcal{Q}$-semimartingales The simplest $\mathcal{Q}$-semimartingales are those for which the structure condition $\mathcal{Q}$ is saturated, i.e. $V_. = \frac{1}{2}\langle M\rangle_.$ or $\underline{V}_. = -\frac{1}{2}\langle M\rangle_.$. Because of their importance, we refer to them as q (resp. $\underline{q}$) semimartingales, and denote them by

$$\begin{cases} r_.(r_0, M) & \equiv r_0 + M_. - \frac{1}{2}\langle M\rangle_. \equiv r_0 + r_.(M), \\ \underline{r}_.(r_0, M) & \equiv \underline{r}_0 + M_. + \frac{1}{2}\langle M\rangle_. \equiv \underline{r}_0 - r_.(-M). \end{cases} \tag{8}$$

The operator $M \to r_.(M)$ is not an additive operator, nevertheless $r_.(M) + r_.(M') = r_.(M + M') + \langle M, M'\rangle_.$ and $r_.(M) - r_.(M') = r_.(M - M') - \langle M - M', M'\rangle_.$.

Taking the exponential of $r_.(M)$ immediately leads to the exponential martingale $\mathcal{E}(M)_. = e^{r_.(M)}$ defined in (7), whilst the exponential of $\underline{r}_.(M)$ leads to $e^{\underline{r}_.(M)} = (\mathcal{E}(-M)_.)^{-1}$.

It will also be interesting to introduce some asymmetry in the previous definition of $\mathcal{Q}$-semimartingales, with the notion of $\mathcal{Q}$-submartingales, especially useful when characterizing the former.

Definition 2.2. *A $\mathcal{Q}$-submartingale is a continuous (or làdlàg)) semimartingale $X_. = X_0 - V_. + M_.$ such that $A_. \equiv -V_. + \frac{1}{2}\langle M\rangle_.$ is a predictable increasing*

process. Equivalently, $e^{X_\cdot} = e^{X_0 + A_\cdot}\mathcal{E}_\cdot(M)$ *is a continuous (làdlàg) submartingale.*

Obviously a $\mathcal{Q}$-semimartingale is a $\mathcal{Q}$-submartingale. Remarkably, applying this property to both X and $-X$ is sufficient to characterize $\mathcal{Q}$-semimartingales. From a financial point of view, this means that the same rules have to be used to characterize both the buyer's and the seller's price.

Theorem 2.3. *Let $X_\cdot$ be a làdlàg optional process. Then, $X_\cdot$ is a $\mathcal{Q}$-semimartingale if and only if both processes X and $-X$ are $\mathcal{Q}$-submartingales, or equivalently if and only if $\exp(X_\cdot)$ and $\exp(-X_\cdot)$ are submartingales. In all cases, $X_\cdot$ is a continuous process.*

Proof. We only have to prove the sufficiency. Assume that $\exp(X_\cdot)$ and $\exp(-X_\cdot)$ are two làdlàg submartingales, with respective multiplicative decomposition $(\overline{M}_\cdot, \overline{A}_\cdot)$, and $(\underline{M}_\cdot, \underline{A}_\cdot)$. Taking the logarithm leads to two different decompositions of X,

$$X_\cdot = X_0 + \overline{M}_\cdot - \tfrac{1}{2}\langle \overline{M}\rangle_\cdot + \overline{A}_\cdot \quad \text{and} \quad -X_\cdot = -X_0 + \underline{M}_\cdot - \tfrac{1}{2}\langle \underline{M}\rangle_\cdot + \underline{A}_\cdot$$

Since the martingales and their quadratic variations are continuous, the jumps of X are the same as the positive jumps of the increasing process $\overline{A}_\cdot$. The same remark holds true for the jumps of the process $-X$. As, the jumps of X are simultaneously positive and negative, the process $X_\cdot$ is continuous.

Moreover, from the uniqueness of the predictable decomposition of $X_\cdot$ we know that $\underline{M}_\cdot = -\overline{M}_\cdot$. Hence, $\langle \underline{M}\rangle = \langle \overline{M}\rangle$, and $\overline{A}_\cdot + \underline{A}_\cdot = \langle M\rangle_\cdot$. From Radon-Nikodym's Theorem, there exists a predictable process $\alpha_\cdot$, with $0 \leqslant \alpha_t \leqslant 2$, such that $d\overline{A}_t = \frac{1}{2}\alpha_t d\langle M\rangle_t$. Substituting $\overline{A}_\cdot$ into the decomposition of $X_\cdot$, we get $dX_t = -\frac{1}{2}(1-\alpha_t)d\langle M\rangle_t + dM_t$ with $|1-\alpha_t| \leqslant 1$. Therefore, $X_\cdot$ is a $\mathcal{Q}$-semimartingale. ☐

Characterization of $\mathcal{Q}(\Lambda, C, \delta)$-semimartingales via exponential transformation In the general structure condition (6), the presence of the term $|Y_\cdot| * C_\cdot$ makes the characterization of quadratic semimartingales more difficult to obtain. Nevertheless the transformations proposed in the following proposition can partially reduce the problem to $\mathcal{Q}$-submartingales.

Theorem 2.4. *Let us introduce the following transformations of any adapted (làdlàg) process $Y_.$:*

$$X_t^{\Lambda,C}(Y) \equiv Y_t + \Lambda_t + \int_0^t |Y_s| dC_s = Y_t + D_t^{\Lambda,C}, \tag{9}$$

$$U_t^{\Lambda,C}(e^Y) \equiv e^{Y_t} + \int_0^t e^{Y_s} d\Lambda_s + \int_0^t e^{Y_s}|Y_s| dC_s. \tag{10}$$

Then, $Y_.$ is a $\mathcal{Q}(\Lambda, C, \delta)$-semimartingale if and only if one of the two equivalent properties is satisfied:
(i) *both processes $X_.^{\Lambda,C}(\delta Y)$ and $X_.^{\Lambda,C}(-\delta Y)$ are $\mathcal{Q}$-submartingales,*
(ii) *both processes $U_t^{\Lambda,C}(e^{\delta Y})$ and $U_t^{\Lambda,C}(e^{-\delta Y})$ are submartingales.*

The link between the two transformations $X^{\Lambda,C}$ and $U^{\Lambda,C}$ is clear when Y is a continuous semimartingale, since $dU_t^{\Lambda,C}(e^Y) = e^{-D_t^{\Lambda,C}} de^{X_t^{\Lambda,C}(Y)}$ (see proof below). The motivation behind the transformation $U_t^{\Lambda,C}(e^Y)$, first introduced by Briand & Hu [7] will be presented later in Section 3.

Proof. We can assume $\delta = 1$ without any loss of generality.
(i) a) Necessary condition: Let $\alpha_.^V \in [-1, 1]$ be a predictable process such that $V_. = \alpha_.^V * (\Lambda_. + |Y| * C_. + \frac{1}{2}\langle M \rangle_.)$. The semimartingale $X_.^{\Lambda,C}(Y) = Y_. + \Lambda_. + |Y| * C_. = Y_. + D_.^{\Lambda,C}(Y)$ is associated with the martingale $M_.$ and the finite variation process $-V_.^X$ where $V_.^X = V_. - D_.^{\Lambda,C}(Y) = (\alpha_.^V - 1) * D_.^{\Lambda,C}(Y) + \frac{1}{2}\alpha_.^V * \langle M \rangle_.$. Since the process $-V_.^X + \frac{1}{2}\langle M \rangle_. = (1 - \alpha_.^V) * \big(D_.^{\Lambda,C}(Y) + \frac{1}{2}\langle M \rangle_.\big)$ is an increasing process, the semimartingale $X_.^{\Lambda,C}(Y)$ is a $\mathcal{Q}$-submartingale.
(i) b) Assume now that both processes $e^{\overline{X}_.}$ and $e^{\underline{X}_.}$ are submartingales, where $\overline{X}_. \equiv X_.^{\Lambda,C}(Y)$ and $\underline{X}_. \equiv X_.^{\Lambda,C}(-Y)$. The processes $\overline{X}_.$ and $\underline{X}_.$ satisfy the following relations:

$$\tfrac{1}{2}(\overline{X}_. - \underline{X}_.) = Y_., \text{ and } \quad \tfrac{1}{2}(\overline{X}_. + \underline{X}_.) = D_.^{\Lambda,C} = \Lambda_. + \tfrac{1}{2}|\overline{X}_. - \underline{X}_.| * C_..$$

Using the same notation and arguments as above, the processes $\overline{X}_.$ and $\underline{X}_.$, whose exponentials are submartingales, can only have positive jumps. This contradicts the fact that their sum is a continuous increasing process. Hence, both processes are continuous. For the same reasons, the sum $\underline{M}_. + \overline{M}_.$ is identically equal to 0, and the sum of increasing processes $\frac{1}{2}(\underline{A}_. + \overline{A}_.) = D_.^{\Lambda,C} + \frac{1}{2}\langle \overline{M} \rangle_. \equiv \frac{1}{2}G_.^{\Lambda,C}$.
There exists a predictable process $\alpha_.$, with $\alpha_. \in [0, 2]$, such that $\overline{A}_. = \frac{1}{2}\alpha_. * G_.^{\Lambda,C}$. Substituting $\overline{A}_.$ in the decomposition of $Y_. = \frac{1}{2}(\overline{X}_. - \underline{X}_.)$, we get $dY_t = -\frac{1}{2}(1 - \alpha_t) dG_t^{\Lambda,C} + d\overline{M}_t$. Therefore, $Y_.$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale.

(ii) a) Let $Y_{\cdot}$ be a $\mathcal{Q}(\Lambda, C)$-semimartingale. Since $X_{\cdot}^{\Lambda,C}(Y) = Y_{\cdot} + D_{\cdot}^{\Lambda,C}$, we have $e^{Y_{\cdot}} = e^{-D_{\cdot}^{\Lambda,C}} e^{X_{\cdot}^{\Lambda,C}(Y)}$. From the classical Itô's formula,

$$de^{Y_t} = e^{-D_t^{\Lambda,C}} de^{X_t^{\Lambda,C}(Y)} - e^{Y_t} dD_t^{\Lambda,C} \quad \text{and} \quad dU_t^{\Lambda,C}(e^Y) = e^{-D_t^{\Lambda,C}} de^{X_t^{\Lambda,C}(Y)}.$$

Then when $Y_{\cdot}$ is a continuous process, $\exp(X_{\cdot}^{\Lambda,C}(Y))$ is a submartingale iff $U_{\cdot}^{\Lambda,C}(e^Y)$ is a submartingale.
(ii) b) Assume now that both processes $U_{\cdot}(e^Y)$ and $U_{\cdot}(e^{-Y})$ are làdlàg submartingales. Let $U_{\cdot}(e^Y) = U_0 + \overline{N}_{\cdot} + \overline{K}_{\cdot}$ and $U_{\cdot}(e^{-Y}) = U_0 + \underline{N}_{\cdot} + \underline{K}_{\cdot}$ be their respective additive decompositions. As before, we can show that the process $Y_{\cdot}$ is continuous. The previous equivalence yields to the result. $\square$

3 Exponential uniform integrability and entropic inequalities

In the previous section, we have obtained a simple characterization of $\mathcal{Q}(\Lambda, C)$-semimartingales using an exponential transformation, leading naturally to positive submartingales defined by their multiplicative or additive decomposition. Whenever submartingales have good integrability properties, the existence of an additive decomposition is equivalent to the submartingale inequalities. It is the famous Doob-Meyer decomposition. The main objective of this section is to precise such integrability properties and the subsequent inequalities.

3.1 Uniform integrability, class $(\mathcal{D})$ and their exponential equivalents

The class $\mathcal{U}_{\exp}$ In the classical martingale theory, uniformly integrable (u.i.) martingales (in particular the conditional expectation of some positive integrable random variable) play a key role as martingale equalities are then valid between two stopping times. The class of such martingales is denoted by $\mathcal{U}$.
In the exponential framework, any exponential martingale $\mathcal{E}(M)_{\cdot}$ of a continuous martingale $M_{\cdot}$ is a positive local martingale, with expectation $\leqslant 1$, hence a supermartingale. The process $\mathcal{E}_{\cdot}(M)$ is a u.i. martingale on $[0, T]$ if and only $\mathcal{E}_t(M) = \mathbb{E}\big[\mathcal{E}_T(M)|\mathcal{F}_t\big]$ $\mathbb{P}$ $a.s.$. It is therefore natural to introduce the class $\mathcal{U}_{\exp}$ of continuous martingales M such that $\mathcal{E}_{\cdot}(M)$ is a uniformly integrable martingale.

The classes $\mathbb{L}^1_{\exp}$ and $(\mathcal{D}_{\exp})$ A $\mathcal{F}_T$-measurable random variable X_T belongs to $\mathbb{L}^1$ provided that $\mathbb{E}(|X_T|) < \infty$ and by definition belongs to $\mathbb{L}^1_{\exp}$ if $\exp(X_T) \in \mathbb{L}^1$.
The optional processes X for which the absolute value is dominated by a uniformly integrable martingale are said to be in the class[2] $(\mathcal{D})$. They are also characterized by the fact that the family of random variables $\{X_\sigma; \sigma \leqslant T, \sigma\,\text{stopping times}\}$ is uniformly integrable. When adopting the exponential point of view, we can extend this notion into:

$X_.$ is said to be in the class $(\mathcal{D}_{\exp})$ if $e^{X_.}$ belongs to the class $(\mathcal{D})$.

Observe that $|X_.|$ belongs to the class $(\mathcal{D}_{\exp})$ if and only if $X_.$ and $-X_.$ belong to the class $(\mathcal{D}_{\exp})$. The sufficient condition is based on the intermediate result that $\cosh(X_.) = \cosh(|X_.|)$ is in the class $(\mathcal{D})$.

$(\mathcal{D})$-submartingales and conditional inequalities A submartingale $S_.$ (as defined in its general form in Subsection 2.2), which is in the class $(\mathcal{D})$, satisfies the following conditional "submartingale inequality"

$$\text{for any stopping times } \sigma \leqslant \tau \leqslant T, \quad S_\sigma \leqslant \mathbb{E}[S_\tau|\mathcal{F}_\sigma],\ a.s..$$

Conversely, it is well-known that any càdlàg process in the class $(\mathcal{D})$ satisfying these inequalities admits a Doob-Meyer decomposition into a martingale and a predictable càdlàg increasing process (see Protter [43], Chapter 3), that is is a submartingale in the previous sense. The less standard làdlàg case, motivated by optimal stopping problems, has been established by Dellacherie et Meyer [15].

3.2 Entropic inequalities and quadratic semimartingales

Entropic submartingales When considering a positive $(\mathcal{D})$-submartingale $S_.$, the logarithm $X_. = \ln S_.$ is a $\mathcal{Q}$-submartingale in the class $(\mathcal{D}_{\exp})$ and satisfies the so-called *entropic inequality*:

$$\forall\, \sigma \leqslant \tau \leqslant T, \quad X_\sigma \leqslant \rho_\sigma(X_\tau)\ a.s. \text{ where } \rho_\sigma(X_\tau) = \ln \mathbb{E}\big[\exp(X_\tau)|\mathcal{F}_\sigma\big]. \quad (11)$$

The operator $\rho_.$ is known as the *entropic process* and has been intensively studied in the framework of risk measures (see for instance Barrieu & El Karoui [5] or [6]). Since conversely, any $\mathcal{Q}$-submartingale in the class $(\mathcal{D}_{\exp})$

[2] P.A. Meyer used the term "class $(\mathcal{D})$", in the honor of J.L.Doob.

satisfies the entropic inequalities, we refer to it as *entropic submartingale.*
An example of entropic submartingale is the simple process $r_{.}(M)$ defined in Equation (8) with $M_{.} \in \mathcal{U}_{\text{exp}}$. In this case, $\exp r_{.}(M) = \mathcal{E}_{.}(M)$ is a positive u.i. martingale, equal to the conditional expectation of its terminal value $\exp(r_T(M))$. Since $\xi_T \equiv r_T(M) \in \mathbb{L}^1_{\text{exp}}$, we can recover $r_t(M)$ from its terminal condition from the following identity[3] based on the entropic process $\rho_{.}(\xi_T)$:

$$r_t(M) = \ln \mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] = \rho_t(\xi_T), \quad \xi_T \equiv r_T(M) \tag{12}$$

The conditional properties of the u.i. martingale $\exp(r_t(M)) = \mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] = \mathbb{E}[\exp(\xi_T)]\mathcal{E}_t(M)$ are translated into the time consistency property of the entropic process over any pair of stopping times (σ, τ) such that $\sigma \leqslant \tau$, $\rho_\sigma(\xi_T) = \rho_\sigma(\rho_\tau(\xi_T))$.
Finally, let us observe that $\rho_{.}(\xi_T)$ is the smallest q-semimartingale $X_{.} = X_0 + r_{.}(N)$ with the terminal value $X_T = \xi_T$. This is a simple consequence of the fact that $\exp(X_{.})$ is a positive local martingale and hence a supermartingale.

Entropic inequalities and $\mathcal{Q}$-semimartingales We are now able to give another formulation for the characterization of $\mathcal{Q}$-semimartingales in the class $(\mathcal{D}_{\text{exp}})$ in terms of inequalities involving the entropic process. This formulation will prove to be better suited than that of Theorem 2.4 when taking limits as we will see in a later section:

Theorem 3.1. *Let $X_{.}$ be a làdlàg optional process such that $|X_T| \in \mathbb{L}^1_{\text{exp}}$. Then $X_{.}$ is a $\mathcal{Q}$-semimartingale such that $|X_{.}| \in (\mathcal{D}_{\text{exp}})$ if and only if $X_{.}$ and $-X_{.}$ are entropic submartingales, or equivalently if for any pair of stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T$,*

$$-\rho_\sigma\big(-X_\tau\big) \equiv \underline{\rho}_\sigma\big(X_\tau\big) \leqslant X_\sigma \leqslant \rho_\sigma\big(X_\tau\big) \quad \mathbb{P}-a.s. \tag{13}$$

Proof. Thanks to Subsection 3.1, when $|X_T| \in \mathbb{L}^1_{\text{exp}}$, the following equivalences hold true: ($X_{.}$ is a $\mathcal{Q}$-semimartingale such that $X_{.}$ and $-X_{.}$ are in the class $(\mathcal{D}_{\text{exp}})$) is equivalent to ($e^{X_{.}}$ and $e^{-X_{.}}$ are $(\mathcal{D})$-submartingales) that is equivalent to ($e^{X_{.}}$ and $e^{-X_{.}}$ satisfy the submartingale inequalities). $\square$

[3]Note that the identity $\rho_t(\xi_T) = r_t(\rho_0(\xi_T), M)$ has suggested the notation $r_t(M)$ for the logarithm of some exponential martingale.

Entropic inequalities and $\mathcal{Q}(\Lambda, C)$-semimartingales The same type of characterization applied to the processes $X_T^{\Lambda,C}(Y)$ or $U_T^{\Lambda,C}(e^Y)$ involves inequalities depending on the process $Y_.$ itself and therefore is often difficult to use. A possible (but not equivalent) way is to work with the process $\bar{X}_.^{\Lambda,C}(|Y|)$ defined as $\bar{X}_t^{\Lambda,C}(|Y|) \equiv e^{C_t}|Y_t| + \int_0^t e^{C_s} d\Lambda_s$ as a generalization of $|Y_.|$ by assuming that the process $\exp(\bar{X}_.^{\Lambda,C}(|Y|))$ is in the class $(\mathcal{D})$.

Proposition 3.2. *Let $\bar{X}_t^{\Lambda,C}(|Y|) \equiv e^{C_t}|Y_t| + \int_0^t e^{C_s} d\Lambda_s$.*
(i) *Let Y be a $\mathcal{Q}(\Lambda, C)$ semimartingale. Then the process $\bar{X}_.^{\Lambda,C}(|Y|)$ is a $\mathcal{Q}$-submartingale.*
(ii) *Let Y be an optional làdlàg process with $\bar{X}_T^{\Lambda,C}(|Y|) \in \mathbb{L}^1_{\exp}$.*
Then the process $\bar{X}_.^{\Lambda,C}(|Y|)$ is an entropic submartingale if and only if the following inequalities hold true for any pair of stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T$, where for $t \leqslant u$, $C_{t,u} \equiv C_u - C_t$,

$$|Y_\sigma| \leqslant \rho_\sigma(e^{C_{\sigma,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{\sigma,t}} d\Lambda_t), \quad \mathbb{P} - a.s. \tag{14}$$

Proof. For the sake of simplicity, we omit Y in $\bar{X}_t^{\Lambda,C}(|Y|)$, and $D_.^{\Lambda,C}(|Y|)$.
(i) By Itô-Tanaka formula involving the sign function ($\mathrm{sign}(x) = x/|x|$), with $\mathrm{sign}(0) = 0$, and the local time $L_.(Y)$ of $Y_.$ at 0, $|Y_.| = |Y_0| + \mathrm{sign}(Y) * Y_. + L_.(Y) = |Y_0| + M_.^s - V_.^s + L_.(Y)$, where $dM_t^s = \mathrm{sign}(Y)_t\, dM_t$ and $dV_t^s = \mathrm{sign}(Y)_t\, dV_t$. This decomposition leads to the following representation of the differential of $\bar{X}_.^{\Lambda,C} = e^{C_.}|Y_.| + e^{C_.} * \Lambda_.$:

$$\begin{aligned} d\bar{X}_t^{\Lambda,C} &= e^{C_t}\big[|Y_t|dC_t + d\Lambda_t + dM_t^s - dV_t^s + dL_t(Y)\big] \\ &= e^{C_t}\big[dD_t^{\Lambda,C} + \tfrac{1}{2}d\langle M\rangle_t - dV_t^s + dL_t(Y)\big] + e^{C_t}\big(dM_t^s - \tfrac{1}{2}d\langle M\rangle_t\big). \end{aligned}$$

Observe that $\bar{A}_.^s = D_.^{\Lambda,C} + \frac{1}{2}\langle M\rangle_. - V_.^s + L_.(Y)$ is an increasing process. The martingale part of $\bar{X}_.^{\Lambda,C}(|Y|)$ is $\bar{M}_.^C = e^{C_.} \star M_.^s$ with quadratic variation $\langle \bar{M}^C\rangle_. = e^{2C_.} * \langle M\rangle_.$. So, the following decomposition shows that $\bar{X}_.^{\Lambda,C}$ is a $\mathcal{Q}$-submartingale since $e^{C_.} - 1 \geqslant 0$,

$$d\bar{X}_.^{\Lambda,C} = e^{C_.}\big[d\bar{A}_.^s + \tfrac{1}{2}(e^{C_.} - 1)d\langle M\rangle_.\big] + d\, r_.(e^{C_.} \star M_.^s).$$

(ii) a) The assumption that $\exp(\bar{X}_.^{\Lambda,C})$ is a $(\mathcal{D})$-submartingale implies in particular that $\bar{X}_T^{\Lambda,C} \in \mathbb{L}^1_{\exp}$, and that $\bar{X}_0^{\Lambda,C} = |Y_0| \leqslant \rho_0(\bar{X}_T^{\Lambda,C})$. The same inequality holds true if we start at time σ with horizon τ by considering the σ-conditional expectation of $\bar{X}_{\sigma,\tau}^{\Lambda,C} = e^{C_{\sigma,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{\sigma,t}} d\Lambda_t$, so that $|Y_\sigma| \leqslant \rho_\sigma(\bar{X}_{\sigma,\tau}^{\Lambda,C})$.
b) Conversely, assume Inequality (14), $|Y_\sigma| \leqslant \rho_\sigma(\bar{X}_{\sigma,\tau}^{\Lambda,C})$. Observe that the

entropic process $\rho_{\delta,t}(\xi_T) = \frac{1}{\delta}\rho_t(\delta\xi_T)$ is increasing with respect to the parameter δ (from the Hölder inequality for the exponential). Then, since $e^{C_\sigma} \geqslant 1$, we have: $\rho_\sigma(e^{C_{\sigma,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{\sigma,t}}d\Lambda_t) \leqslant e^{-C_\sigma}\rho_\sigma(e^{C_{0,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{0,t}}d\Lambda_t)$. So $\bar{X}^{\Lambda,C}_. = e^{C_.}|Y_.| + e^{C} * \Lambda_.$ satisfies the entropic inequalities $\bar{X}^{\Lambda,C}_\sigma \leqslant \rho_\sigma\big(e^{C_{0,\tau}}|Y_\tau| + \int_0^{\dot{\tau}} e^{C_{0,t}}d\Lambda_t + \int_0^\tau e^{C_{0,t}}d\Lambda_t\big) = \rho_\sigma(\bar{X}^{\Lambda,C}_\tau)$. Taking $\tau = T$, it follows that $\bar{X}^{\Lambda,C}_.$ is dominated by the $(\mathcal{D}_{\exp})$-process $\rho_.(\bar{X}^{\Lambda,C}_T)$ and so is an entropic submartingale. Hence the result. □

The properties of the dominating process $\rho_.(e^{C_{.,T}}|Y_T| + \int_.^T e^{C_{.,s}}d\Lambda_s)$ are therefore essential to obtain results for the process $Y_.$. The non-adapted process $\phi_{.,T}(|Y_T|) = e^{C_{.,T}}|Y_T| + \int_.^T e^{C_{.,s}}d\Lambda_s$ with initial condition $\phi_{0,T}(|Y_T|) = \bar{X}^{\Lambda,C}_T(|Y|)$, first introduced in Briand & Hu [7] (Lemma 1), is the positive decreasing solution of the ordinary differential equation, with terminal condition $|Y_T|$,

$$d\phi_t = -(d\Lambda_t + |\phi_t|dC_t), \quad \phi_T = |Y_T|. \tag{15}$$

In order words, the non-adapted process $U^{\Lambda,C}_.(e^{\phi_{.,T}}) = e^{\phi_{.,T}} + \int_0^. e^{\phi_{s,T}}d\Lambda_s + \int_0^. e^{\phi_{s,T}}|\phi_{s,T}|dC_s$ is constant and equal to $e^{\phi_{0,T}}$. This property is the main motivation for introducing the $U^{\Lambda,C}$ transformation.
The decreasing property of $\exp(\phi_{.,T})$ explains the supermartingale property of the process $\Phi_.(|Y_T|)$ defined as the optional projection of $\exp(\phi_{.,T})$:

$$\Phi_\sigma(|Y_T|) \equiv \mathbb{E}[\exp(\phi_{\sigma,T}(|Y_T|))|\mathcal{F}_\sigma] = \exp\big(\rho_\sigma(e^{C_{\sigma,T}}|Y_T| + \int_\sigma^T e^{C_{\sigma,t}}d\Lambda_t)\big). \tag{16}$$

Note that, for the sake of clarity, we often omit the reference to Y_T in $\phi_{.,T}(|Y_T|)$, $\Phi_.(|Y_T|)$ or $\bar{X}^{\Lambda,C}_T(|Y_T|)$.

Theorem 3.3. *Assume* $\mathbb{E}[\exp(\bar{X}^{\Lambda,C}_T(|Y_T|))] = \mathbb{E}[\exp(\phi_{0,T})] < \infty$.
(i) *The process $\Phi_.$ is a $(\mathcal{D})$-supermartingale dominated by the martingale $\mathbb{E}[e^{\phi_{0,T}}|\mathcal{F}_t] = N^0_t$, with the additive decomposition $\Phi_. = \Phi_0 + N^\Phi_. - A^\Phi_.$. The predictable increasing process is $A^\Phi_. = \int_0^. \Phi_s d\Lambda_s + \int_0^. \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s]dC_s$, when the process $N^\Phi_.$ is a uniformly integrable martingale.*
(ii) *The process $U^{\Lambda,C}_.(\Phi) = \Phi_. + \int_0^. \Phi_s d\Lambda_s + \int_0^. \Phi_s \ln(\Phi_s)dC_s$ is a positive $(\mathcal{D})$-supermartingale, associated with the same u.i. martingale $N^\Phi_.$, and the increasing process $A^U_. = \int_0^.(\mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s] - \Phi_s \ln(\Phi_s))dC_s$.*
(iii) *Assume Inequality (14) for the process $|Y_.|$. The processes $U^{\Lambda,C}_.(e^Y)$ and $U^{\Lambda,C}_.(e^{-Y})$ are two $(\mathcal{D})$-submartingales dominated by the $(\mathcal{D})$-supermartingale $U^{\Lambda,C}_.(\Phi)$.*

Remark 1. *The positive quantity* $H^{\text{ent}}_s(e^{\phi_{s,T}}) \equiv \mathbb{E}[e^{\phi_{s,T}}\phi_{s,T}|\mathcal{F}_s] - \Phi_s \ln(\Phi_s)$ *appearing in* $A^U_.$ *is well-known in statistics as the conditional Shannon entropy of the random variable* $e^{\phi_{s,T}}$. *Its properties will be studied in the next subsection when considering integrability properties of the supremum.*

Proof. As observed by Briand & Hu [7] (Lemma 1), since $\phi_{t,T}$ is a positive solution of the differential equation $d\phi_t = -(d\Lambda_t + \phi_t dC_t)$, the non-adapted process $U^{\Lambda,C}_t(e^{\phi_{.,T}})$ is constant, $U^{\Lambda,C}_t(e^{\phi_{.,T}}) = e^{\phi_{t,T}} + \int_0^t e^{\phi_{s,T}} d\Lambda_s + \int_0^t e^{\phi_{s,T}}|\phi_{s,T}|dC_s = e^{\phi_{0,T}}$, with $\phi_{0,T} = \bar{X}^{\Lambda,C}_T \in \mathbb{L}^1_{\text{exp}}$. The dynamics of the supermartingale $\Phi_t = \mathbb{E}[e^{\phi_{t,T}}|\mathcal{F}_t]$ is obtained by taking conditional expectation in this relation.

(i) First observe that the assumption $e^{\phi_{0,T}} \in \mathbb{L}^1$ implies that $e^{\phi_{T,T}} \in \mathbb{L}^1$ and that the non-adapted increasing process $B^\phi_t = \int_0^t e^{\phi_{s,T}} d\Lambda_s + \int_0^t e^{\phi_{s,T}}|\phi_{s,T}|dC_s$ is integrable. Since $\Phi_.$ is the optional projection of $e^{\phi_{.,T}}$, and since both increasing processes $\Lambda_.$ and $C_.$ are adapted, the dual predictable projection of B^ϕ_t is the continuous process $A^\Phi_t = \int_0^t \Phi_s d\Lambda_s + \int_0^t \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s]dC_s$, generating the same conditional variation, $\mathbb{E}[B^\phi_{t,T} - A^\Phi_{t,T}|\mathcal{F}_t] = 0$. So the process $N^1_t = \mathbb{E}[B^\phi_T - A^\Phi_T|\mathcal{F}_t] = \mathbb{E}[B^\phi_t - A^\Phi_t|\mathcal{F}_t]$ is a uniformly integrable martingale. Then, taking the conditional expectation of the constant process $U^{C,\Lambda}_.(e^{\phi_{.,T}})$ implies that $\Phi_t + A^\Phi_t + N^1_t = N^0_t$, and $N^\Phi_t = N^0_t - N^1_t$.

(ii) To show that $U^{\Lambda,C}_.(\Phi)$ is also a supermartingale, we use that the Shannon entropy (see Remark 3.3) $H^{\text{ent}}_s(e^{\phi_{s,T}}) = \mathbb{E}[e^{\phi_{s,T}}|\phi_{s,T}||\mathcal{F}_s] - \Phi_s \ln(\Phi_s)$ is positive, and the process $A^U_. = \int_0^. H^{\text{ent}}_s(e^{\phi_{s,T}})dC_s$ is increasing. Then, some simple calculation shows that $U^{\Lambda,C}_t(\Phi) + A^U_t = \Phi_t + A^\Phi_t = N^\Phi_t$ is a positive u.i martingale, that provides the Doob-Meyer decomposition of the supermartingale $U^{\Lambda,C}_.(\Phi)$.

(iii) This last statement is a straightforward consequence of the inequality $e^{|Y_.|} \leqslant \Phi_.$. ☐

Remark 2. *The key condition to obtain these properties is that the process* $U^{\Lambda,C}_.(\Phi(|Y_T|))$ *is a* $(\mathcal{D})$*-supermartingale. Note that this is also true if we replace* $|Y_T|$ *by any* $\mathcal{F}_T$*-random variable* $|\eta_T| \geqslant |Y_T|$, *such that* $e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s \in \mathbb{L}^1_{\text{exp}}$.

Remark 3. *As observed by Briand & Hu [7], extending the results of Lepeltier & San Martin in [33], the linear growth condition in* $Y_.$, $|Y_.| * C_.$, *may replaced by a superlinear growth* $h(|Y_.|) * C_.$, *where* h *is an increasing convex* C^1 *function, with* $h(0) > 0$, *satisfying the integrability condition* $\int_0^T du \frac{|u|}{h(u)} = +\infty$.

The function $\phi(t)$ is then replaced by the solution of the ODE $\phi'(t) = -h(\phi_t)$ with a terminal condition $\phi(T) = z \geqslant 0$.

Maximal exponential integrability and $L \log L$-condition When[4] looking for entropic inequalities, assuming that the exponential of the processes is in the class $(\mathcal{D})$ is a minimal assumption. However, it is sometimes interesting to obtain estimates on the exponential of the maximum of these processes. Entropic inequalities reduce the problem to the estimation of the running supremum of some entropic processes, or equivalently to the running supremum of some positive martingales, for which we can apply standard Burkholder-Davis-Gundy (BDG) martingale inequalities. An excellent presentation of the different martingale inequalities may be found in Lenglart, Lepingle & Pratelli [31].

From now on, we adopt the following non-standard notation for the running supremum of some measurable process X: $\max|X_t| = \max_{0\leqslant u\leqslant t}|X_u|$ and $\max|X_{s,t}| = \max_{s\leqslant u\leqslant t}|X_u - X_s|$. The space of semimartingales $X_.$ such that $\max|X_T| \in \mathbb{L}^p$ $(p \geqslant 1)$ is denoted by $\mathcal{S}^p$. For continuous local martingales, the relevant quantity is the quadratic variation and we denote by $\mathbb{H}^p$ the space of martingales with a quadratic variation in $\mathbb{L}^p$. Moreover, for any continuous local martingale $M_.$, such that $M_0 = 0$, the BDG inequality gives some estimates of its maximum in terms of with its quadratic variation as, for any $0 < p < \infty$, there exist two positive constants c_p and C_p such that:

$$\text{for any } 0 < p < \infty,\ c_p\mathbb{E}[\langle M\rangle_T^{p/2}] \leqslant \mathbb{E}[\max|M|_T^p] \leqslant C_p\mathbb{E}[\langle M\rangle_T^{p/2}].$$

The following Doob inequalities, based on the terminal condition and only true for $p > 1$, are more classical:

$$\text{for any } p > 1,\ k_p\,\mathbb{E}[\langle M\rangle_T^{p/2}] \leqslant \mathbb{E}[|M|_T^p] \leqslant K_p\,\mathbb{E}[\langle M\rangle_T^{p/2}].$$

So, for $p > 1$, $\mathbb{E}[\max|M|_T^p] < \infty$ if and only $\mathbb{E}[\langle M\rangle_T^{p/2}] < \infty$. In other words, the spaces $\mathcal{S}^p$ and $\mathbb{H}^p$ coincide.

In terms of exponential martingale $L_. = \mathcal{E}_.(M)$, these results become:

$$\forall\, p > 1,\ L_. \in \mathcal{S}^p \Longleftrightarrow L_T = \exp(r_T(M) \in \mathbb{L}^p \Longleftrightarrow (\textstyle\int_0^T L_s^2 d\langle M\rangle_s)^{1/2} \in \mathbb{L}^p.$$

When $p < 1$, a similar maximal inequality holds true for exponential martingales or more generally for positive supermartingales [31],

$$\forall\, p < 1 \quad \mathbb{E}[\max L_T^p] \leqslant \tfrac{\mathbb{E}((L_0)^p)}{1-p}.$$

[4]This paragraph can be omitted for a first reading

When $p = 1$ and the local martingale is positive, we have to use the following $L \log L$- condition:

Proposition 3.4. *Let $L_. = \exp(M_. - \frac{1}{2}\langle M\rangle_.)$ be a positive continuous locale martingale and $\max L_t$ its running supremum.*
(i) *(Doob) Assume that $L_.$ is a u.i. martingale.*

Then $$\mathbb{E}(\max L_T) - 1 = \mathbb{E}\big(L_T \ln(\max L_T)\big) \geqslant \mathbb{E}\big(L_T \ \ln(L_T)\big)$$

and $$\mathbb{E}\big(L_T \ \ln(L_T)\big) = \mathbb{E}\big(L_T \, \tfrac{1}{2}\langle M\rangle_T\big).$$

(ii) *(Harremoës) The following inequality is sharp:*

$$\mathbb{E}(\max L_T) - 1 - \ln(\mathbb{E}(\max L_T)) \leqslant \mathbb{E}\big(L_T \ \ln(L_T)\big) = H^{\mathrm{ent}}(L_T). \tag{17}$$

The martingale $L_.$ belongs to $\mathcal{S}^1$ if and only if $\mathbb{E}\big(L_T \ \ln(L_T)\big) < \infty$.
(iii) *Let $U_.$ be a positive $(\mathcal{D})$-submartingale with deterministic initial condition U_0 and $m = \mathbb{E}(U_T) \geqslant U_0$. The previous Harremoës inequality becomes, when $u_m(x) = x - m - m\ln(x)$,*

$$u_m\big(\mathbb{E}(\max U_T)\big) - u_m(U_0) \leqslant \mathbb{E}\big(U_T \ \ln(U_T)\big) - \mathbb{E}(U_T) \ln\big(\mathbb{E}(U_T)\big) = H^{\mathrm{ent}}(U_T).$$

In particular, $\mathbb{E}(\max U_T)$ is dominated by an increasing function of $H^{\mathrm{ent}}(U_T) + u_m(U_0)$.

Proof. The proof is based on Dellacherie [14] and Harremoës [25].
(i) Since $L_.$ is a continuous process, $\max L_.$ only increases on the set $\{L_. = \max L_.\}$ and $\max L_t = 1 + \int_0^t d\max L_s = 1 + \int_0^t \frac{L_s}{\max L_s} d\max L_s$. Taking the expectation (after stopping at some stopping time bounding $\max L_.$ on $[0,T]$ if necessary) and using the fact that $L_.$ is the conditional expectation of its terminal value leads to $\mathbb{E}\big(\max L_T\big) - 1 = \mathbb{E}\big(L_T \ln(\max L_T)\big)$.
(i) a) Since $\ln(\max L_T) \geqslant \ln(L_T)^+$, and $L_t \ \ln(L_t)^- \leqslant 1/e$, then $|L_T \ln(L_T)| \in \mathbb{L}^1$ when $\max L_T \in \mathbb{L}^1$. This establishes the necessary condition.
(ii) To prove that finite entropy implies integrability of the max, we show Inequality (17). We start by studying $\mathbb{E}\big(L_T \ln(\max L_T)\big) - \mathbb{E}\big(L_T \ln(L_T)\big)$ from the concavity of the function ln. Given that $x^* = \mathbb{E}(\max L_T) = \mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)$ if $\mathbb{Q} = L_T.\mathbb{P}$, $\mathbb{E}_{\mathbb{Q}}\big(\ln(\max L_T/L_T)\big) \leqslant \ln\big(\mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)\big) = \ln x^*$. Inequality (17) is then easily obtained. An example of càdlàg martingale satisfying the equality may be found in Harremoës [25].

(iii) The extension to $U_.$ being a positive submartingale does not present any specific difficulties other than purely computational, , since $\mathbb{E}\big(\max U_T\big) - U_0 \leqslant \mathbb{E}\big(U_T \ln(\max U_T/U_0)\big)$. Taking now $\mathbb{Q} = (U_T/m).\mathbb{P}$, $x^*/m = \mathbb{E}(\max U_T)/m =$

$\mathbb{E}_{\mathbb{Q}}(\max L_T/L_T)$, the convexity inequality becomes: $\mathbb{E}_{\mathbb{Q}}(\ln(\max U_T/U_T)) \leqslant \ln\big(\mathbb{E}_{\mathbb{Q}}(\max U_T/U_T)\big) = \ln(x^*/m)$. Some elementary algebra gives the final result. Observe that u_m is convex and minimal at $z = m$. Since $m_0 \leqslant m$, $u_m(U_0) \geqslant u_m(m)$. Then since the entropy is positive, $u_m(U_0) + H^{\text{ent}}(U_T)$ belongs to the range of $\{u_m(z); z \geqslant m\}$ and $\mathbb{E}\big(\max U_T\big) \leqslant u_m^{-1}(u_m(U_0) + H^{\text{ent}}(U_T))$.

(i) b) We now show the link between entropy and quadratic variation. Assume that $L_T \ln(L_T) \in \mathbb{L}^1$. Let T_K be an increasing sequence of stopping times, such that $\ln(L_t) = M_t - \frac{1}{2}\langle M\rangle_t$ is bounded by K. The sequence T_K is increasing and goes to infinity with K. Thanks to the Girsanov theorem, $N^{\mathbb{Q}}_{.} = M_{.} - \langle M\rangle_{.}$ is a local martingale with respect to the probability measure $\mathbb{Q} = L_T.\mathbb{P}$, and $\mathbb{E}\big(L_T \frac{1}{2}\langle M\rangle_T\big) = \lim_K \mathbb{E}\big(L_T \frac{1}{2}\langle M\rangle_{T\wedge T_K}\big) = \lim_K \mathbb{E}\big(L_{T\wedge T_K} \frac{1}{2}\langle M\rangle_{T\wedge T_K}\big)$. Using $\mathbb{E}\big(L_{T\wedge T_K} N^{\mathbb{Q}}_{T\wedge T_K}\big) = 0$,

$$\begin{aligned}
&\mathbb{E}\big(L_{T\wedge T_K} \tfrac{1}{2}\langle M\rangle_{T\wedge T_K}\big) = \mathbb{E}\big(L_{T\wedge T_K}(M_{T\wedge T_K} - \langle M\rangle_{T\wedge T_K} + \tfrac{1}{2}\langle M\rangle_{T\wedge T_K})\big)\\
&= \mathbb{E}\big(L_{T\wedge T_K} \ln(L_{T\wedge T_K})\big) \leqslant \mathbb{E}\big(\max L_{T\wedge T_K}\big) - 1 \leqslant \mathbb{E}\big(\max L_T\big) - 1.
\end{aligned}$$

Then $N^{\mathbb{Q}}_{.}$ is a square integrable $\mathbb{Q}$-martingale and $\mathbb{E}_{\mathbb{Q}}(\ln(L_T)) = \mathbb{E}_{\mathbb{Q}}(\frac{1}{2}\langle M\rangle_T)$, which is is the desired equality. □

Let us now come back to the question of maximal inequalities for $\mathcal{Q}(\Lambda, C)$-semimartingales. The various results are based on the behaviour of the entropic process $\rho_{.}(\bar{X}^{\Lambda,C}_T(|Y_T|))$ also denoted $\rho_{.}(\bar{X}^{\Lambda,C}_T)$. To give a concise form to the various but similar estimates, we introduce the following family of positive increasing functions ψ_p defined on $\mathbb{R}^+$ by $\psi_p(z) = z^p$ if $p \neq 1$ and $\psi_1(z) = z\ln z - z + 1$. Note that, as in the previous subsections, we consider separately the case of entropic submartingales.

Proposition 3.5. (i) *Assume $X_{.} = X_0 - V_{.} + M_{.}$ to be an entropic submartingale ($|X_T| \in \mathbb{L}^1_{\exp}$), such that $\psi_p(\exp X_T) \in \mathbb{L}^1$ provided that $p \geqslant 1$. Then, both processes $\exp(X_{.})$ and $\mathcal{E}(M)_{.}$ belong to $\mathcal{S}^p$, and their $\mathcal{S}^p$ norm are dominated by some increasing function of $\mathbb{E}(\psi_p(\exp X_T))$ for $p \geqslant 1$, and of $\psi_p(\mathbb{E}(\exp X_T))$ for $p < 1$.*

(ii) *Let $Y_{.}$ be a $\mathcal{Q}(\Lambda, C)$-semimartingale such that $\psi_p(\exp \bar{X}^{\Lambda,C}_T) \in \mathbb{L}^1$ when $p \geqslant 1$. The processes $\exp(\rho_{.}(\bar{X}^{\Lambda,C}_T))$, $\Phi_{.}(|Y_T|)$, $\exp(e^{C_{.}}|Y_{.}| + \int_0^{.} e^{C_s} d\Lambda_s)$ and $\mathcal{E}(e^C * M)_{.}$ belong to $\mathcal{S}^p$ and their $\mathcal{S}^p$ norms are dominated by some increasing function of $\mathbb{E}(\psi_p(\exp \bar{X}^{\Lambda,C}_T))$ for $p \geqslant 1$ or $\psi_p(\mathbb{E}(\exp \bar{X}^{\Lambda,C}_T))$ for $p < 1$.*

Proof. (i) The proof relies on the multiplicative decomposition of the sub-

martingale $\exp(X_.) = \exp(X_0 + A_.)\mathcal{E}(M)_.$. Then $\exp(X_.)$ and $\mathcal{E}(M)_.$ have the same maximal properties. The proof is a simple consequence of the entropic inequalities (3.2), BDG inequalities and the maximal estimates given in Proposition 3.4;
(ii) The maximal estimates of $\exp(\rho_.(\bar{X}_T^{\Lambda,C}))$ are a simple consequence of (i) , and yield to the other estimates since the different process are dominated by $\exp(\rho_.(\bar{X}_T^{\Lambda,C}))$. For the process $\mathcal{E}(e^C \star M)_.$, we have to use the decomposition of the entropic submartingale $e^{C_.}Y_. + \int_0^. e^{C_s} d\Lambda_s$. $\square$

Change of probability measures and entropy Let L be a positive local martingale with $L_0 = 1$. The condition $\mathbb{E}\big(L_T \ln(L_T)\big) = H^{\text{ent}}(L) < \infty$ naturally appears when considering the martingale L as the likelihood of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, as it measures the positive Shannon entropy $H^{\text{ent}}(d\mathbb{Q}/d\mathbb{P}) = \mathbb{E}\big(d\mathbb{Q}/d\mathbb{P} \ln(d\mathbb{Q}/d\mathbb{P})\big)$ of $\mathbb{Q}$ with respect to $\mathbb{P}$. The previous result states that $H^{\text{ent}}(d\mathbb{Q}/d\mathbb{P})$ is finite if and only if the martingale density $L_.$ is in $\mathcal{S}^1$.
This interpretation is particularly interesting when using the variational formulation of the the entropic risk measure $\rho_0(\xi_T)$ (see for instance, Frittelli [23], Föllmer & Schied [22])

$$\rho_0(\xi_T) = \sup_{\mathbb{Q}}\{\mathbb{E}_{\mathbb{Q}}(\xi_T) - H^{\text{ent}}(\mathbb{Q}/\mathbb{P})|\, H(\mathbb{Q}/\mathbb{P}) < +\infty\}. \tag{18}$$

In other words, when $\xi_T \in \mathbb{L}^1_{\text{exp}}$, for any martingale density $L^Q \in \mathcal{S}^1$ whose the S^1 norm is bounded by K, we have an uniform estimate of $\mathbb{E}_Q(\xi_T)$ given by $\mathbb{E}_Q(\xi_T) \leqslant \rho_0(\xi_T) + \mathbb{E}(L_T \ln(L_T)) \leqslant \rho_0(\xi_T) + K$.
Moreover, when the random variable ξ_T itself is associated with a finite relative entropy probability measure $\mathbb{Q}^{\xi_T}$ defined by its density $L_T^{\xi_T} = e^{(\xi_T - \rho_0(\xi_T))}$, we can prove by a simple verification that the supremum is attained for $\mathbb{Q}^{\xi_T}$.
Very recently, Choulli & Schweizer [10] have developed applications to mathematical finance of the $L \log L$ condition.

4 Quadratic variation estimates and stability results

We are now capable to establish the main contribution of this paper, i.e. some stability results, which require some uniform estimation of key quantities, including quadratic variation and running supremum. In order to use

the previous inequalities, we need the family of $\mathcal{Q}(\Lambda, C)$-semimartingales we consider to be uniformly dominated. Following Remark 2, we can replaced Y_T by a generic random variable η_T such that $|\eta_T| \geqslant |Y_T|$ and $\bar{X}_T^{C,\Lambda}(|\eta_T|)$ satisfies an appropriate integrability condition. Therefore, it seems natural to introduce the following class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$, and to work within this class of quadratic semimartingales:

Definition 4.1. *Let $|\eta_T|$ be a $\mathcal{F}_T$-random variable, such that $\bar{X}_T^{C,\Lambda}(|\eta_T|) = e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s$ belongs to $\mathbb{L}^1_{\exp}$. The class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ is the set of $\mathcal{Q}(\Lambda, C)$-semimartingales $Y_.$ such that $|Y_.| \leqslant \rho_.(e^{C_{.,T}}|\eta_T| + \int_.^T e^{C_{.,s}} d\Lambda_s)$ a.s..*

4.1 Quadratic variation estimates

We now study the quadratic variation of $\mathcal{Q}(\Lambda, C)$-semimartingale $Y_.$ when $Y_.$ belongs to $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$. Following Kobylanski [30], the best way to do so is to use the function $v(x) = e^x - 1 - x$ instead of the simple exponential function. This function is indeed positive, convex, and increasing for $x \geqslant 0$, and satifies $v''(x) - v'(x) = 1$. In the following, we use the short notation $\bar{X}_T^{C,\Lambda}(|\eta_T|) = \bar{X}_T^{C,\Lambda}$.

Theorem 4.2 (Quadratic variation estimates)**.** *Let $Y_. \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$.*
(i) *Then, the quadratic variation $\langle M\rangle_.$ of the $\mathcal{Q}(\Lambda, C)$-semimartingale $Y_. = Y_0 + M_. - V_.$ satisfies for any stopping times $\sigma \leqslant T$,*

$$\frac{1}{2}\mathbb{E}[\langle M\rangle_{\sigma,T}|\mathcal{F}_\sigma] \leqslant \Phi_\sigma(|Y_T|)\mathbf{1}_{\{\sigma<T\}} \leqslant \mathbb{E}[\exp(\bar{X}_T^{C,\Lambda}(|\eta_T|))\,\mathbf{1}_{\{\sigma<T\}}|\mathcal{F}_\sigma]. \quad (19)$$

In particular, the martingale $M_.$ is in $\mathbb{H}^2$, with the uniform estimate

$$\mathbb{E}[\frac{1}{2}\langle M\rangle_T] \leqslant \mathbb{E}[\exp(\bar{X}_T^{C,\Lambda}(|\eta_T|))]. \quad (20)$$

(ii) *Let $p^\eta = \sup\{p;\ \mathbb{E}\big[\exp(p\bar{X}_T^{C,\Lambda}(|\eta_T|))\big] < +\infty\}$. Then $p^\eta \geqslant 1$ and $\forall p \in [1, p^\eta[$, the martingale M belongs to $\mathbb{H}^{2p}$, and*

$$\mathbb{E}[\langle M\rangle_T^p] \leqslant (2p)^p\,\mathbb{E}\big[\exp(p\bar{X}_T^{C,\Lambda}(|\eta_T|))\big]. \quad (21)$$

(iii) *If $\Phi_t(|\eta_T|) = \mathbb{E}[\exp(e^{C_{t,T}}|\eta_T| + \int_t^T e^{C_{t,u}} d\Lambda_u)|\mathcal{F}_t]$ is uniformly bounded in $t \leqslant T$, then the conditional quadratic variation $\frac{1}{2}\mathbb{E}[\langle M\rangle_{\sigma,T}|\mathcal{F}_\sigma]$ is uniformly bounded. Hence $M_.$ is a BMO-martingale.*

Proof. By analogy with the previous notation, when using the function $v(x) = e^x - 1 - x$, we set $V_t^{\Lambda,C}(e^{|Y|}) = v(|Y_t|) + \int_0^t v'(|Y_s|)(d\Lambda_s + |Y_s|dC_s) = v(|Y_t|) + \int_0^t v'(|Y_s|)dD_s^{\Lambda,C}$. So, $U_t^{\Lambda,C}(e^{|Y|}) - V_t^{\Lambda,C}(e^{|Y|}) = 1 + |Y_t| + D_t^{\Lambda,C}(|Y|)$, and both processes $U_.^{\Lambda,C}$ and $V_.^{\Lambda,C}$ are in the class $(\mathcal{D})$ since $Y_. \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$.
(i) 1) As we see in the proof of Proposition 3.2, the semimartingale $|Y_.|$ is associated with the martingale $M_.^s = \operatorname{sign}(Y_.) \star M_.$, the finite variation process $V_.^s = \operatorname{sign}(Y_.) * V_.$ and the local time at $\{0\}$, that disappears in the Itô's formula since $v'(0) = 0$. Using similar calculation to those of the previous section, and the identity $v''(x) - 1 = v'(x)$, we obtain that the process $V_t^{\Lambda,C}(e^{|Y|}) - \frac{1}{2}\langle M\rangle_t = v(|Y_0|) + \int_0^t v'(|Y_s|)dM_s^s + \int_0^t v'(|Y_s|)(dD_s^{\Lambda,C} - dV_s^s + \frac{1}{2}d\langle M\rangle_s)$ is a submartingale, and since $V_.^{\Lambda,C}$ is in the class $(\mathcal{D})$, for any $\sigma \leqslant T$, $\mathbb{E}[\frac{1}{2}\langle M\rangle_{\sigma,T}|\mathcal{F}_\sigma] \leqslant \mathbb{E}[v(|Y_T|) - v(|Y_\sigma|) + \int_\sigma^T v'(|Y_s|)dD_s^{\Lambda,C}|\mathcal{F}_\sigma]$.

2) Since, by definition, $\forall x \geqslant 0, 0 \leqslant v(x) \leqslant e^x$ and $v'(x) \leqslant e^x$,

$$\int_\sigma^T v'(|Y_s|)dD_s^{\Lambda,C} \leqslant \int_\sigma^T \Phi_s(d\Lambda_s + \ln|\Phi_s|dC_s) \text{ for any } \sigma \leqslant T.$$

Thanks to the supermartingale property of $U_.^{\Lambda,C}(\Phi)$ (Proposition 3.3 (ii)) and the inequality $\Phi_. \geqslant \exp(|Y_.|)$ (implying in particular $\Phi_T = \exp(|\eta|) \geqslant v(|Y_T|)$), we have $\mathbb{E}[\int_\sigma^T \Phi_s(d\Lambda_s + \ln|\Phi_s|dC_s)|\mathcal{F}_\sigma] \leqslant \mathbb{E}[\Phi_\sigma - \Phi_T|\mathcal{F}_\sigma]$ and

$$\begin{aligned}
&\mathbb{E}[\tfrac{1}{2}\langle M\rangle_{\sigma,T}|\mathcal{F}_\sigma] \leqslant \mathbb{E}[v(|Y_T|) - v(|Y_\sigma|) - (\Phi_T - \Phi_\sigma)|\mathcal{F}_S]\\
&= \mathbb{E}\big[\big(-(\Phi_T - v(|Y_T|) + v(|Y_\sigma|)) + \Phi_\sigma\big)\mathbf{1}_{\{\sigma<T\}}|\mathcal{F}_\sigma\big]\\
&\leqslant \Phi_\sigma \mathbf{1}_{\{\sigma<T\}} \leqslant \mathbb{E}[\exp \bar{X}_T^{\Lambda,C}\,\mathbf{1}_{\{\sigma<T\}}|\mathcal{F}_\sigma].
\end{aligned}$$

(ii) As observed in Lenglart, Lépingle & Pratelli [31], the final result is a simple consequence of the so-called Garsia-Neveu Lemma (Lemma 4.3) (see for instance Neveu [41]) recalled below.
(iii) This is a straightforward consequence of the inequality $\mathbb{E}[\frac{1}{2}\langle M\rangle_{\sigma,T}|\mathcal{F}_\sigma] \leqslant \Phi_\sigma(|\eta_T|)$. □

Lemma 4.3 (Garsia-Neveu Lemma)**.** *Let $A_.$ be a predictable càdlàg increasing process and U a random variable, positive and integrable. If for any stopping times $\sigma \leqslant T$, $\mathbb{E}[A_T - A_\sigma|\mathcal{F}_\sigma] \leqslant \mathbb{E}[U\mathbf{1}_{\{\sigma<T\}}|\mathcal{F}_\sigma]$,*

$$\forall p \geqslant 1, \quad \mathbb{E}[A_T^p] \leqslant p^p\mathbb{E}[U^p].$$

More generally, $\mathbb{E}[F(A_T)] \leqslant \mathbb{E}[F(pU)]$ for any convex function F such that $p = \sup_{x>0}(x(\ln F)'(x)) < +\infty$

Here we apply this lemma to the random variable $U = \exp\big(\bar{X}_T^{\Lambda,C}(|\eta_T|)\big)$ for any $p \geqslant 1$ such that $U \in \mathbb{L}^p$. As a corollary of this result, uniform estimates may be obtained for the total variation of the process $V_.$:

Corollary 4.4. *Let $Y_. \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$. The total variation of the process $V_.$ such that $Y_. = Y_0 + M_. - V_.$ satisfies for $1 \leqslant p < p^\eta$*

$$\mathbb{E}[|V|_T^p] \leqslant (2p)^p \, \mathbb{E}\big[\exp(p\bar{X}_T^{C,\Lambda})\big], \tag{22}$$

When $\Phi_.(|\eta_T|) = \mathbb{E}[\exp(e^{C_{.,T}}|\eta_T| + \int_.^T e^{C_{t,u}} d\Lambda_u)|\mathcal{F}_.]$ is bounded by K_C, then $\mathbb{E}[|V|_{\sigma,T}|\mathcal{F}_\sigma] \leqslant 2K_C$.

Proof. Since $V_.$ satisfies the structure condition $\mathcal{Q}(\Lambda, C)$, $\mathbb{E}[|V|_{\sigma,T}|\mathcal{F}_\sigma] \leqslant E[\Lambda_{\sigma,T} + \int_\sigma^T |Y_s| dC_s + \frac{1}{2}\langle M\rangle_{\sigma,T}|\mathcal{F}_\sigma] \leqslant 2\mathbb{E}[\exp(\bar{X}_T^{\Lambda,C})\mathbf{1}_{\{\sigma<T\}}|\mathcal{F}_\sigma]$. Indeed, $\mathbb{E}[\Lambda_{\sigma,T} + \int_\sigma^T |Y_s| dC_s|\mathcal{F}_\sigma] \leqslant \mathbb{E}[\int_\sigma^T e^{|Y_s|}(d\Lambda_s + |Y_s|dC_s)|\mathcal{F}_\sigma] \leqslant \mathbb{E}[(\Phi_\sigma - \Phi_T)|\mathcal{F}_\sigma] \leqslant \mathbb{E}[\exp(\bar{X}_T^{\Lambda,C})\mathbf{1}_{\{\sigma<T\}}|\mathcal{F}_\sigma]$. We conclude with Lemma 4.3. □

4.2 Stability results for $\mathcal{Q}(\Lambda, C)$-semimartingales

We can start by noticing that the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ is stable by a.s. convergence, since the submartingale property of both processes $U_.(e^Y)$ and $U_.(e^{-Y})$, dominated by the $(\mathcal{D})$-supermartingale $U_.(\Phi)$, is stable by a.s. convergence. Moreover, Theorem 2.4 implies that the limit process is continuous and is also in $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$. However, previous estimates of both quadratic variation and finite variation processes suggest that a better stability result may hold true, in particular regarding the strong convergence of the martingale parts. The space of martingales, where this convergence takes place, depends essentially on the exponential integrability properties of the random variable $X_T^{\Lambda,C}(|\eta_T|)$. The method is very similar to that of Lepeltier & San Martin [32]. When the $\mathcal{Q}(\Lambda, C)$-semimartingales are bounded, this type of results has already been obtained for the $\mathbb{H}^2$-convergence by Kobylanski [30] and Morlais [39]. Our stability result is novel and direct, and gives better convergence results with the $\mathbb{H}^1$ convergence. This result, that appears here for the first time in a BSDE framework, is based on an old result of Barlow & Protter [4] on the convergence of semimartingales.

Theorem 4.5. *Assume the sequence $(Y_.^n)$ of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales is a Cauchy sequence for the a.s. uniform convergence, i.e. $\sup_{t\leqslant T}|Y_t^n - Y_t^{n+p}|$ tends to 0 almost surely when $n \to \infty$.*
Then the limit of the sequence of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-semimartingales $(Y_.^n)$ is a $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-semimartingale $Y_. = Y_0 + M_. - V_.$.
Different types of convergence hold true for the processes $(M_.^n, V_.^n)$ of the

decomposition $Y^n_. = Y^n_0 + M^n_. - V^n_.$:

(i) *Martingales convergence of* $(M^n_.)$ *to* $M_.$.

 a) *The sequence* $(M^n_.)$ *converges to* $M_.$ *in* $\mathbb{H}^1$.

 b) *If for some* $p > 1$ $\bar{X}^{\Lambda,C}_T(|\eta_T|) \in \mathbb{L}^p_{\exp}$, *the sequence* $(M^n_.)$ *converges to* $M_.$ *in* $\mathbb{H}^{2p}$, *and in the BMO-space if* $\Phi_S(|\eta_T|)$ *is bounded.*

(ii) *The sequence of finite variation processes* $(V^n_.)$ *converges at least in* $\mathcal{S}^1$ *to the process* $V_.$ *satisfying the structure condition* $\mathcal{Q}(\Lambda, C)$.

Proof. We proceed[5] in several steps to prove this convergence result. We first introduce some notations and make some elementary calculations. For $s \leqslant t$, let $Y^{i,j}_t = Y^i_t - Y^j_t$, $\quad M^{i,j}_t = M^i_t - M^j_t$ and $Y^{i,j}_{s,t} = (Y^i_t - Y^i_s) - (Y^j_t - Y^j_s)$, and the short notation first introduced in Subsection 3.2, $\sup_{s \leqslant u \leqslant t} | Y^{i,j}_u - Y^{i,j}_s| \equiv \max | Y^{i,j}_{s,t}|$. Then for any stopping times $\sigma \leqslant \tau \leqslant T$,

$$\begin{aligned} \langle M^{i,j}\rangle_{\sigma,\tau} &= |Y^{i,j}_{\sigma,\tau}|^2 - 2\int_\sigma^\tau Y^{i,j}_{\sigma,s}\, dY^{i,j}_s \\ &\leqslant |Y^{i,j}_{\sigma,\tau}|^2 - 2\int_\sigma^\tau Y^{i,j}_{\sigma,s}\, dM^{i,j}_s + 2\int_\sigma^\tau |Y^{i,j}_{\sigma,s}|\, d(|V^j|_s + |V^i|_s) \end{aligned}$$

Using either the fact that $Y^{i,j}$ is bounded, or a uniform localization procedure, the stochastic integral $\int_\sigma^{\tau_n} Y^{i,j}_{\sigma,s}\, dM^{i,j}_s$ has null conditional expectation for a well-chosen stopping time τ_n. Then, thanks to the monotonicity of $\langle M\rangle$ and Corollary 4.4, with $B^{i,j} = 2(|V^i| + |V^j|)$,

$$\begin{aligned} \mathbb{E}\big[\langle M^{i,j}\rangle_{\sigma,T} \mid \mathcal{F}_\sigma\big] &\leqslant \mathbb{E}\big[\max |Y^{i,j}_{\sigma,T}|^2\, \mathbf{1}_{\{\sigma<T\}}| + \int_\sigma^T \max |Y^{i,j}_{\sigma,s}|\, dB^{i,j}_s \mid \mathcal{F}_\sigma\big] \\ &\leqslant \mathbb{E}\big[(\max |Y^{i,j}_{0,T}|^2 + \max |Y^{i,j}_{0,T}| B^{i,j}_T)\, \mathbf{1}_{\{\sigma<T\}}| \mathcal{F}_\sigma\big]. \end{aligned}$$

We now start with the proof corresponding to the assumption $\bar{X}^{\Lambda,C}_T(|\eta_T|) \in \mathbb{L}^p_{\exp}$, since it is very similar to the proof in the linear growth case (see Lepeltier & San Martin [32]).

(i) b) Thanks to the Garsia-Neveu Lemma (Lemma 4.3), for $r \geqslant 1$,

$$\begin{aligned} &\mathbb{E}\big[\langle M^{i,j}\rangle^r_T\,\big] \leqslant r^r \mathbb{E}\big[(\max |Y^{i,j}_{0,T}|^2 + \max |Y^{i,j}_{0,T}|\; B^{i,j}_T)^r\big] \\ &\leqslant \frac{1}{2}(2r)^r\{\mathbb{E}\big[(\max |Y^{i,j}_{0,T}|)^{2r}\big] + \mathbb{E}\big[(\max |Y^{i,j}_{0,T}| B^{i,j}_T)^r\big]\}. \end{aligned}$$

Then, since $B^{i,j}_T$ belongs to $\mathbb{L}^p$, by Hölder inequalities, for any p and q such

[5] An earlier proof of this result in the BMO case is due to Nicolas Cazanave, a former PhD student at Ecole Polytechnique

that $\frac{1}{p}+\frac{1}{q}=1$, and $1 \leqslant r < p$, if $K_r = \frac{1}{2}(2r)^r$

$$\begin{aligned}
&\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|B^{i,j}_T)^r\big] \leqslant (\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|)^q\big])^{\frac{r}{q}}(\mathbb{E}\big[(B^{i,j}_T)^p\big])^{\frac{r}{p}}\\
&\mathbb{E}\big[\langle M^{i,j}\rangle^r_T\big] \leqslant K_r\{\mathbb{E}\big[\max|Y^{i,j}_{0,T}|^{2r}\big] + (\mathbb{E}\big[(\max|Y^{i,j}_{0,T}|)^q\big])^{\frac{r}{q}}(\mathbb{E}\big[(B^{i,j}_T)^p\big])^{\frac{r}{p}}\}.
\end{aligned}$$

From the monotonicity of both sides of this inequality with respect to r, we can take $r = p$. We have used that $\max|Y^{i,j}_{0,T}|$ has finite moments of all orders since as shown in Paragraph 3.2 $\max|Y^i_{0,T}|$ and $\max|Y^i_{0,T}|$ are in $\mathbb{L}^p_{\text{exp}}$. Hence, we have the desired convergence.

(i) c) In the bounded case, thanks to Corollary 4.4, the conditional total variation $\mathbb{E}[|V^n|_{\sigma,T}|\mathcal{F}_\sigma]$ are uniformly bounded by C_V. To obtain the BMO convergence, we have to modify the previous proof, by using an integration by parts formula involving the conditional variation of $B^{i,j}$,

$$\begin{aligned}
\mathbb{E}[\textstyle\int_\sigma^T \max|\,Y^{i,j}_{\sigma,s}|\,dB^{i,j}_s\mid\mathcal{F}_\sigma] \;&= \mathbb{E}[\textstyle\int_\sigma^T d_u\max|\,Y^{i,j}_{\sigma,u}|\,\Big(\mathbb{E}[\int_u^T dB^{i,j}_s\mid\mathcal{F}_u]\Big)\mid\mathcal{F}_\sigma]\\
&\leqslant 2\,C_V\mathbb{E}[\max|\,Y^{i,j}_{\sigma,T}|\mid\mathcal{F}_\sigma].
\end{aligned}$$

and so $\quad \mathbb{E}[\langle M^{i,j}\rangle_{\sigma,T}\mid\mathcal{F}_\sigma] \leqslant 2\,C_V\mathbb{E}[\max|\,Y^{i,j}_{\sigma,T}|\mid\mathcal{F}_\sigma] + \mathbb{E}\big[|Y^{i,j}_{\sigma,T}|^2\mid\mathcal{F}_\sigma\big].$

Then, the BMO-convergence holds true.

(i) a) The proof of the general case requires a different argument, based on a result of Barlow & Protter [4] on the convergence of semimartingales. In the framework of quadratic semimartingales, the key points are the uniform estimates of both the quadratic variation and the total variation given in Theorem 4.2, Equation (20) and Corollary 4.4. The proof given in [4] of the $\mathbb{H}^1$-convergence of the martingales is based on the square root of the inequality given at the beginning of the proof,

$$\langle M^{i,j}\rangle_t \leqslant |Y^{i,j}_{0,t}|^2 - 2\textstyle\int_0^t Y^{i,j}_{0,s}\,dM^{i,j}_s + 2\int_0^t |Y^{i,j}_{0,s}|\,dB^{i,j}_s.$$

The first step is to estimate the square root of $\max|Y^{i,j}_{0,.}\star M^{i,j}_{0,T}|$ using the Burkolder-Davis-Gundy inequalities for continuous martingales for $p=\frac{1}{2}$, that have been recalled in Subsection 3.2:

$\mathbb{E}[\max|Y^{i,j}_{0,.}\star M^{i,j}_T|^{\frac{1}{2}}] \leqslant \bar{C}\mathbb{E}[\langle Y^{i,j}_{0,.}\star M^{i,j}\rangle^{1/4}_T]$ where $\bar{C}$ is a universal constant.

Then, since $\mathbb{E}[\langle Y^{i,j}_{0,.}\star M^{i,j}\rangle^{1/4}_T] \leqslant \mathbb{E}[(\max|Y^{i,j}_{0,T}|)^{\frac{1}{2}}\langle M^{i,j}\rangle^{1/4}_T]$,

$$\begin{aligned}
\mathbb{E}\big[\sqrt{\langle M^{i,j}\rangle_T}\big] \;&\leqslant \mathbb{E}[\max|Y^{i,j}_{0,T}|] + \sqrt{2}\bar{C}\,\mathbb{E}\big[\max|Y^{i,j}_{0,T}|\big]^{\frac{1}{2}}\mathbb{E}\big[\sqrt{\langle M^{i,j}\rangle_T}\big]^{\frac{1}{2}}\\
&+\sqrt{2}\mathbb{E}[\max|Y^{i,j}_{0,T}|]^{\frac{1}{2}}\mathbb{E}[B^{i,j}_T|]^{\frac{1}{2}}.
\end{aligned}$$

Since $\mathbb{E}\big[\sqrt{\langle M^{i,j}\rangle_T}\big]$ and $\mathbb{E}[B_T^{i,j}]$ are uniformly bounded, and $\mathbb{E}[\max|Y_T^{i,j}|]$ goes to 0, then $\mathbb{E}\big[\sqrt{\langle M^{i,j}\rangle_T}\big]$ also goes to 0. The $\mathbb{H}^1$-convergence of the martingale part is established.

(ii) The next point is to study the convergence of the sequence $(V_.^n)$ to a process $V_.$ satisfying the same structure condition $\mathcal{Q}(\Lambda, C)$. Since, the sequence $(Y_.^n,\, M_.^n, \langle M^n\rangle_.^{\frac{1}{2}})$ converges in $\mathcal{S}^1$ to $(Y_.,\, M_., \langle M\rangle_.^{\frac{1}{2}})$, the sequence $(V_.^n)$ also converges in $\mathcal{S}^1$. Therefore, we can extract a subsequence, still denoted $(Y_.^n, M_.^n, V_.^n,\, \langle M^n\rangle_.^{\frac{1}{2}})$, such that the sequence converges uniformly in time almost surely.

(iii) This point is obvious since, as observed at the beginning of this section, the class $\mathcal{S}(|\eta_T|)$ is stable by a.s. convergence. $\square$

Stability results for BSDE-like quadratic semimartingales To obtain the convergence of the finite variation processes in total variation, we need to make additional assumption on the processes V^n, as in the BSDE framework. We adopt the general setting where the reference to the Brownian framework is relaxed as in El Karoui & Huang [19].

Definition 4.6 (BSDE-like quadratic semimartingale). *Let us consider a continuous predictable increasing process $K_.$, a d-dimensional continuous orthogonal martingale $N_. = (N_.^i)_{i=1}^d$, with quadratic variation $\langle N^i\rangle_.$ strongly dominated by $K_.$ such that $d\langle N^i\rangle_t = \gamma_t^i dK_t$, two increasing processes $\Lambda_.$ and $C_.$ also dominated by $K_.$ such that $d\Lambda_t = l_t dK_t$ and $dC_t = c_t dK_t$, such that all processes $\gamma_.^i, l_., c_.$ are bounded by k (for instance, $K_. = \sum_{i=1}^d \langle N^i\rangle_. + \Lambda_. + C_.$ and $k = 1$.) The coefficient $g(.,y,z)$ is a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}\times\mathbb{R}^d)$ measurable process, often assumed to be continuous with respect to (y,z).*
A semimartingale $Y_.$ with the decomposition $Y_. = Y_0 - V_. + M_.$ is said to have a quadratic coefficient g if $dY_t = -dV_t + dM_t$, with

$$\begin{cases} dV_t = g(t,Y_t,Z_t)\,dK_t,\;\; dM_t = Z_t dN_t + dM_t^{\perp},\;\; \forall i\;\; d\langle N^i, M^{\perp}\rangle_t = 0 \\ |g(t,Y_t,Z_t)| \leqslant \frac{1}{\delta}l_t + |Y_t|c_t + \frac{\delta}{2}|\sqrt{\gamma}_t Z_t|^2, \quad |\sqrt{\gamma}_t Z_t|^2 = \sum_{i=1}^d \gamma_t^i |Z_t^i|^2. \end{cases} \tag{23}$$

The local martingale $Z\star N$ is the orthogonal projection of the local martingale $M_.$ onto the space of stochastic integrals generated by the local martingale $N_.$, and $d\langle Z\star N\rangle_t \ll d\langle M\rangle_t$, so that $d|V|_t \ll \frac{1}{\delta}d\Lambda_t + |Y_t|dC_t + \delta d\langle M\rangle_t$ and $Y_.$ is a quadratic semimartingale.

When considering sequences of BSDE-like quadratic semimartingales under mild assumptions on the sequence of coefficients, the sequence of finite

variation processes is converging in total variation in the appropriate space, and the limit is still a BSDE-like quadratic semimartingale.
The uniform convergence of the quadratic semimartingales needed for these convergence results may seem very strong. We know however from Theorem 2.4 that all the processes obtained by a.s. convergence are continuous. Thanks to Dini's Theorem, the monotone convergence implies uniform convergence for continuous functions on compact spaces. Therefore, by a localization procedure, we can prove the following very strong result:

Theorem 4.7. *Let assume the sequence $(Y^n_.)$ to be a monotone sequence of $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-semimartingales converging almost surely to a process $Y_.$.*
(i) *Then, the limit process $Y_.$ is a continuous $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$- semimartingale, the convergence is locally uniform and all properties given in Theorem 4.5 hold (locally) true. In particular, there exists a subsequence of martingales $M^n_. = Z^n\star N_. + M^{n,\perp}_.$ converging in $\mathbb{H}^1$ and almost surely to $M_. = Z\star N_. + M^\perp_.$*
(ii) *Suppose in addition that the processes $(Y^n_.)$ are BSDE-like quadratic semimartingales, associated with a sequence of monotone coefficients g_n converging almost surely to g, having the following properties:*

a) *The monotone sequence g_n have uniform quadratic growth:*
$$|g_n(t,Y^n_t,Z^n_t)| \leqslant \tfrac{1}{\delta}l_t + |Y^n_t|c_t + \tfrac{\delta}{2}|\sqrt{\gamma}_t Z^n_t|^2,\ d\mathbb{P}\times dK_. a.s.$$
b) *the sequence $g_n(.,Y^n_.,Z^n_.)$ converges to $g(.,Y_.,Z_.)$, $d\mathbb{P}\times dK_. a.s.$.*

Then, the limit process $Y_.$ is a BSDE-like $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-semimartingale with coefficient $g(t,y,z) = \lim g_n(t,y,z)$.

Proof. Note the characterization of $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-semimartingales given in Theorem 2.4 passes to the limit, since all processes $U^{\Lambda,C}_.(e^{|Y^n|})$ are dominated by the $(\mathcal{D})$-process $U^{\Lambda,C}_.(\Phi(|\eta_T|))$. The limit process Y is a continuous $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$-semimartingale, with decomposition $Y_. = Y_0 + M_. - V_.$.
(i) The localization procedure is based on the family (T_K) of stopping times as to bound the u.i martingale $N^0_t = \mathbb{E}[\exp(\phi_0(|\eta_T|)|\mathcal{F}_t]$ by K. By the characterization of u.i. continuous martingale (see for instance Azema, Gundy & Yor [2]), the sequence T_K goes to ∞ and for $K \geqslant K_\epsilon$ large enough, $\mathbb{P}(T_K < T) \leqslant \frac{\epsilon}{K}$. Therefore, the sequence $(Y^n_{.\wedge T_K})$ lives on a compact set where the monotone convergence to a *continuous process* is uniform. The sequence of martingales $(M^n_{.\wedge T_K})_n$ strongly converges in the appropriate space to the martingale $M_{.\wedge T_K}$. The same property holds true for the sequence $V^n_{.\wedge T_K}$. Thanks to the previous estimates, for all these processes $Y^n_., M^n_., V^n_.$ the convergence is uniform on $[0, T\wedge T_K]$ in probability.

(ii) Let $Z^{n,K}_t \equiv Z^n_t \mathbf{1}_{\{t\leqslant T_K\}}$ in such way that $(Z^n \star N)_{.\wedge T_K} = Z^{n,K} \star N_.$. Since the sequence $(M^n_{.\wedge T_K})_n$ strongly converges, the sequences of orthogonal martingales $(M^{n,\perp}_{.\wedge T_K})_n$ and $(Z^{n,K} \star N_.)_n$ also strongly converge in the appropriate space, and at least in $\mathbb{H}^1$.
Therefore, we can extract a subsequence still denoted $Z^{n,K}_.$ converging a.s.. By assumption, for $t \leqslant T_K$ the sequence $g^n(t, Y^n_t, Z^{n,K}_t)$ goes to $g(t, Y_t, Z_t)$ $dK_. \otimes d\mathbb{P}$ *a.s.*. It now remains to show that the convergence is also true in expectation. Observe that $\mathbb{E}[\int_0^{T_K} |g_n(s, Y^n_s, Z^n)_s - g(s, Y_s, Z_s)|\mathbf{1}_{\{|Z^n_s|\leqslant C\}} dK_s]$ goes to 0, by dominated convergence, since $\Phi_.$ and $Y^n_.$ are bounded on $[0, T_K]$. Moreover, since the sequence in n of the quadratic variations at time T_K, $\langle Z^{n,K} \star N\rangle_{T_K}$ is bounded in $\mathbb{L}^1$, for $s \leqslant T_K$, $|g_n(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)| \leqslant \Psi_s + \frac{1}{2}|Z^n_s|^2$, with $\Psi_t \mathbf{1}_{\{t\leqslant T_K\}} \in \mathbb{L}^1(d\mathbb{P}\otimes dK_s)$ and $\mathbb{P}(|Z^n_s| \geqslant C) \leqslant \frac{1}{C^2}\mathbb{E}(|Z^n_s|^2)$. Hence, $\mathbb{E}[\int_0^{T_K} |g_n(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)|\mathbf{1}_{\{|Z^n_s|>C\}} dK_s]$ goes to 0 when C goes to ∞, uniformly in n. As a consequence, the process $V_.$ in the decomposition of the quadratic semimartingale $Y_.$ is given by $dV_t = g(t, Y_t, Z_t)dK_t$ on $[0, T_K]$ for any K. □

Remark 4. *Delbaen, Hu & Bao show in [12] that increasing the growth of the coefficient into a superquadratic growth yields to ill-posed problems. In particular, monotone stability does not hold any more. For classical BSDEs, when the coefficient simply depends on z, superquadratic growth means that* $\limsup g(z)/|z|^2 = \infty$.

5 Existence result for quadratic BSDEs

The question of existence of bounded solutions for the classical quadratic BSDEs in Brownian framework has been solved by Kobylanski [30], using an exponential transformation as to come back to the standard framework of a coefficient with linear growth. A detailed review of the literature including the comparison theorem and different applications may be found in El Karoui, Hamadène & Matoussi [18]. Most of the recent papers focusing on financial applications of quadratic BSDEs consider the situation where the martingale $M_.$ is BMO (see for instance the recent papers by Hu, Imkeller & Muller [27], Ankrichner, Imkeller & Reis [1], [40], or the PhD thesis of Dos Reis [16]). From Theorem 4.2, such a framework is equivalent to look at bounded solutions. Briand & Hu [7] have been the first to extend the previous results to unbounded solutions. In all these papers, as in Kobylanski

[30], the main difficulty is however to prove the strong convergence of the martingale part.
The stability result we have obtained in the previous section opens a new possible direction to tackle this question. The idea is to approximate monotonically the coefficient itself by coefficients with a linear and quadratic growth, for which there are some results on the existence of solution but also for which it is possible to take the limit thanks to the stability Theorem 4.7. In our approach, we do not need this BMO framework and have a stability result prevailing in a wider context, moving away from the bounded case to the case where the terminal condition has exponential moment. Indeed, having bounded solutions is naturally replaced by belonging to the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ as in the previous section, which reduces to an exponential moment condition for $|\eta_T|$, when Λ and $C \equiv 0$. Recall that this last condition is equivalent to have the absolute value of the solution in the class $(\mathcal{D}_{\exp})$ when the coefficient does not depend on y (and $g(t,0,0) \equiv 0$).

We start this section by looking more closely at the interrelationship between quadratic BSDEs and quadratic semimartingales, when the quadratic structure condition is saturated.

5.1 A canonical example: q_δ-BSDE and entropic process

We are focusing on simplest quadratic BSDEs when the structure condition is saturated and the coefficient is simply denoted by q_δ. This framework has a particular importance in finance as it corresponds to that of indifference pricing in incomplete markets when using an exponential utility criterion (in general in the bounded case) as in Rouge & El Karoui [44], and many other papers (see for instance Mania & Schweizer [35]) or the recent book on indifference pricing edited by Carmona [9].
In this simple framework, it is interesting to consider the various possible points of view. In particular, note that the two following problems coincide in a Brownian framework:
(i) First, finding a quadratic q_δ-semimartingale $Y_t = Y_0 + M_t - \frac{\delta}{2}\langle M\rangle_t$ with terminal condition $Y_T = \xi_T$. We refer to the solution as a GBSDE(q_δ, ξ_T)-solution, where G stands for "generalized". The process $-Y$ is a GBSDE-solution associated with $(\underline{q}_\delta, -\xi_T)$.
(ii) In the second case, corresponding to the BSDE general framework (Def-

inition 4.6), the problem is to find $(Y_., M_. \equiv Z * N_. + M^{\perp}_.)$, such that $dY_t = -\frac{\delta}{2}|\sqrt{\gamma_t}Z_t|^2 dK_t - Z_t dN_t - dM^{\perp}_t$ with terminal condition $Y_T = \xi_T$. The similar equation with the opposite process will be also considered.
Based on the previous results, we will consider these two questions in parallel in the paragraphs below.

Summary of previous results on GBSDEs The entropic process $\rho_t(\xi_T)$ defined earlier in Equation (3.2) as $\ln \mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] \equiv \rho_t(\xi_T)$ appears naturally when studying such $(q, \text{or } \underline{q})$-GBSDEs. Indeed, as presented in the following proposition, if the terminal condition $\xi_T \in \mathbb{L}^1_{\exp}$, then $\rho_.(\xi_T)$ is a $(\mathcal{D}_{\exp})$-solution of q-GBSDE. The stronger assumption on the terminal condition $|\xi_T| \in \mathbb{L}^1_{\exp}$ is used for the estimates of the quadratic variation or for some stability result.

Proposition 5.1. (i) *Assume that $\xi_T \in \mathbb{L}^1_{\exp}$. Then the entropic process $\rho_.(\xi_T)$ is the unique $(\mathcal{D}_{\exp})$-solution of the quadratic* GBSDE(q, ξ_T)*, i.e. there exists a martingale $M^{\rho}_. \in \mathcal{U}_{\exp}$ such that*

$$d\rho_t(\xi_T) = -\tfrac{1}{2}d\langle M^{\rho}\rangle_t + dM^{\rho}_t, \quad \rho_T(\xi_T) = \xi_T.$$

Moreover, $\rho_.(\xi_T)$ is minimal in the class of solutions $Y_.$: $\rho_.(\xi_T) \leqslant Y_.$.
(ii) *Assume that $-\xi_T \in \mathbb{L}^1_{\exp}$. The negative entropic process $\underline{\rho}_.(\xi_T)$ is* **a** *solution of the* GBSDE$(\underline{q}, \xi_T)$*, i.e. there exists a martingale $\underline{M}^{\rho}$ such that*

$$d\underline{\rho}_t(\xi_T) = \tfrac{1}{2}d\langle \underline{M}^{\rho}\rangle_t + d\underline{M}^{\rho}_t \quad \underline{\rho}_T(\xi_T) = \xi_T.$$

but in general $\underline{\rho}_.(\xi_T)$ is not a $(\mathcal{D}_{\exp})$-solution.
(iii) *When $|\xi_T| \in \mathbb{L}^1_{\exp}$, then* a)*$\underline{\rho}_t(\xi_T)$ is the maximal solution of the* GBSDE$(\underline{q}, \xi_T)$.
b) *The martingales M^{ρ} and $\underline{M}^{\rho}$ are in $\mathbb{H}^2$ and if ξ_T is bounded, they are BMO-martingales.*
c) *If in addition $|\xi_T| + \ln(|\xi_T|) \in \mathbb{L}^1_{\exp}$, the r.v.* $\max|\rho_{0,T}(\xi_T)|$ *and* $\max|\underline{\rho}_{0,T}(\xi_T)|$ *belong to $\mathbb{L}^1_{\exp}$. Moreover, the following variational representation holds true:*

$$\rho_t(\xi_T) = \operatorname*{ess\,sup}_{M^{\mathbb{Q}}}\{\mathbb{E}_{\mathbb{Q}}\big(\xi_T - \frac{1}{2}\langle M^{\mathbb{Q}}\rangle_{t,T}|\mathcal{F}_t|\,\mathbb{E}_{\mathbb{Q}}(\langle M\rangle^{\mathbb{Q}}_{t,T}) < +\infty\big)\}. \tag{24}$$

Proof. (i) From Section 3 and as $\rho_.(\xi_T) = \rho_0(\xi_T) + r_.(M)$, $\rho_.(\xi_T)$ is the unique $(\mathcal{D}_{\exp})$-solution for the GBSDE(q, ξ_T), and the smallest in the class of the q-semimartingale with the same terminal value.
(ii) Since $-\xi_T \in \mathbb{L}^1_{\exp}$, the process $\rho_.(-\xi_T)$ is well-defined in $(\mathcal{D}_{\exp})$ and $-\rho_.(-\xi_T)$ is solution of the $\underline{q}$-GBSDE, but not in general in the class $(\mathcal{D}_{\exp})$.

(iii) Assume both variables ξ_T and $-\xi_T$ in $\mathbb{L}^1_{\exp}$. Using the convexity of ρ, its follows that $0 = \rho_.(0) \leqslant \frac{1}{2}(\rho_.(\xi_T) - \underline{\rho}_.(\xi_T))$. Then, $\rho_.(\xi_T) \in (\mathcal{D}_{\exp})$ implies $\underline{\rho}_.(\xi_T) \in (\mathcal{D}_{\exp})$.
The comparison with the other solutions is a simple consequence of the fact that $-Y$ is a solution of GBSDE$(q, -\xi_T)$, and therefore bigger than $\rho_.(-\xi_T) = -\underline{\rho}_.(\xi_T)$. The rest of (iii) is a straightforward consequence of Theorem 4.2.
□

For lack of space, we will not further develop the variational point of view, but this approach can be extended to q-BSDEs, using in particular approximations based on the solutions of convex BSDEs with linear growth (see for instance El Karoui, Hamadène & Matoussi [18]).

$(q,$ or $\underline{q})$-BSDEs The question of the existence of solutions of the $(q, \text{or}\ \underline{q})$-BSDEs is more delicate to tackle and does not admit explicit representation. These difficulties also appear in the Brownian framework when the vector martingale N is defined from a limited number of components of the generating Brownian motion. Different methods can be used, the first one is based on linear growth approximating solutions, whilst the second one uses the convexity of the coefficient and represents solutions as value function of some optimization problems. We now develop the first point of view.
In this case, the approximation is based on the coefficients $q_n(z) \equiv \frac{1}{2}(|z|^2 - (z-n)^{+2}) = \frac{1}{2}(|z|^2 \mathbf{1}_{\{|z|\leqslant n\}} + (n|z| - \frac{1}{2}n^2)\mathbf{1}_{\{|z|>n\}})$ with linear and quadratic growth, increasing to $q(z)$ when n goes to infinity. For $\xi_T \in \mathbb{L}^2$, using by the classical theory, the BSDE(q_n, ξ_T) has a unique solution in $\mathcal{S}^2$, bounded if ξ_T is bounded.

Proposition 5.2. *Let $|\eta_T| \in \mathbb{L}^1_{\exp}$, and (ξ^n_T) a sequence of increasing r.v., bounded by $|\eta_T|$ and converging a.s. to ξ_T.*
(i) *Denote by $(Y^n_., Z^n_., M^{n,\perp})_. \in \mathbb{H}^2(\mathbb{R}^+) \otimes \mathbb{H}^2(\mathbb{R}^n)$ the unique solution of the BSDE (q_n, ξ_T). The process $Y^n_.$ is a $\mathcal{Q}$-semimartingale satisfying the entropic inequality $|Y^n_.| \leqslant \rho_.(|\eta_T|)$.*
(ii) *The sequence $(q^n, Y^n_., Z^n_., M^{n,\perp})$ satisfies the hypothesis of Theorem 4.7 and strongly converges to $(Y_., Z_., M^\perp)$, minimal solution of BSDE(q, ξ_T) such that $|Y_.| \leqslant \rho_.(|\eta_T|)$, with the variational representation*

$$Y_t(\xi_T) = \operatorname*{ess\,sup}_\nu \{\mathbb{E}_{\mathbb{Q}^\nu}\big(\xi_T - \frac{1}{2}\int_t^T (|\sqrt{\gamma_s}\nu_s|^2 dK_s|\mathcal{F}_t)\} \tag{25}$$

where $\mathbb{Q}^{\nu}$ is the probability with density $\mathcal{E}(\nu \star N)_{.}$ with finite entropy ($\mathbb{E}_{\mathbb{Q}^{\nu}}(\int_0^T (|\sqrt{\gamma_t}\nu_t|^2 dK_t.) < +\infty$).
(iii) *Uniqueness holds in the class of solutions Y such that $|Y_{.}| \leqslant \rho_{.}(|\xi_T|)$, and $|\xi_T|$ is bounded or such that $\rho_\delta(|\xi_T|)$ for any $\delta > 0$.*

Proof. (i)Its is clear that $Y^n_{.}$ is a $\mathcal{Q}$-semimartingale, bounded if ξ_T is bounded. Then $|Y^n_{.}|$ belongs to the class $(\mathcal{D}_{\exp})$ and satisfies the entropic inequality $|Y^n_{.}| \leqslant \rho_{.}(|\xi_T|)$. Since both processes $Y^n_{.}(\xi_T)$ and $\rho_{.}(\xi_T)$ are monotone with respect to their terminal condition, by approximating $\xi_T)$ by bounded random variables, the entropic inequality holds at the limit under the assumption $|\xi_T| \in \mathbb{L}^1_{\exp}$
(ii) The first result is a direct consequence of the stability result given in Theorem 4.7. The variational representation for Y^n is as in (25) with the restriction that ν is bounded by n. That is a standard result on convex BSDEs with uniformly linear growth (see for instance El Karoui, Peng & Quenez [20] or Theorem 8.7 in El Karoui, Hamadène & Matoussi [18]). Thanks to entropy result in Subsection 3.2, the representation (25) pass to the limit, since ξ_T is $\mathbb{Q}^{\nu}$-integrable.
(iii) Let Y be a solution satisfying $|Y_{.}| \leqslant \rho_{.}(|\xi_T|)$. We first assume that $|\xi_T|$ is bounded, so that all solutions are bounded and the associated martingales $M_{.}$, $M^{\perp}_{.}$ and $M_{.} - M^{\perp}_{.}$ are BMO-martingales.
Denote by $Y^i_{.}$ and $Y^i_{.}$ two solutions satisfying the entropic inequalities with two bounded terminal conditions. Using the same notation than in the proof of Theorem 4.5, we observe that the difference $Y^{i,j}_{.} \equiv Y^i_{.} - Y^j_{.}$ verifies a linear BSDE, with linear growth condition with respect to another probability measure,

$$
\begin{aligned}
dY^{i,j}_t &= -\tfrac{1}{2}\big(|\sqrt{\gamma_t}Z^i_t|^2 - |\sqrt{\gamma_t}Z^j_t|^2\big)dK_t + dM^{i,j}_t \\
&= -\tfrac{1}{2}\big(\sqrt{\gamma_t}Z^{i,j}_t.\sqrt{\gamma_t}(Z^i_t + Z^j_t)\big)dK_t + Z^{i,j}_t.dN_t + dM^{i,j,\perp}_t \\
&= Z^{i,j}_t(dN_t - \tfrac{1}{2}\sqrt{\gamma_t}(Z^i_t + Z^j_t)dK_t) + dM^{i,j,\perp}_t
\end{aligned}
$$

Since $Y^i_{.}$ and $Y^j_{.}$ are bounded solutions, by Theorem 4.2, the martingales $M^i_{.}$, and $M^j_{.}$ are BMO-martingales, implying that the quadratic variation of $\frac{1}{2}(Z^i_{.} + Z^j_{.}) \star N_{.}$ is also conditionally bounded, and then $\frac{1}{2}(Z^1_{.} + Z^2_{.}) \star N_{.}$ is a BMO-martingale. By Girsanov Theorem, $\mathcal{E}(\frac{1}{2}(Z^1_{.} + Z^2_{.}) \star N)_{.}$ is a u.i. exponential martingale defining a new probability measure $\mathbb{Q}^{(i+j)}$ such that $dN^{(i+j)}_t \equiv dN_t - \frac{1}{2}\sqrt{(}Z^i_t + Z^j_t)dK_t)$ is a $\mathbb{Q}^{(i+j)}$-local martingale with the same quadratic variation as $N_{.}$. Moreover, $M^{i,j,\perp}_{.}$ is still a $\mathbb{Q}^{(i+j)}$-local martingale,

orthogonal to $N^{(i+j)}_{\cdot}$. Then, $Y^{i,j}_{\cdot}$ is a bounded $\mathbb{Q}^{(i+j)}$ local martingale, and so a true martingale and $Y^{i,j}_{\cdot} = \mathbb{E}_{\mathbb{Q}^{(i+j)}}(Y^{i,j}_T|\mathcal{F}_{\cdot})$. Uniqueness and comparison theorem are easily deduced of this property.
In the general case the difficulty is to show directly that $\mathcal{E}(\frac{1}{2}(Z^1_{\cdot} + Z^2_{\cdot}) \star N)_{\cdot}$ is u.i. martingale, given that $\mathcal{E}(M^i)_{\cdot}$ and $\mathcal{E}(M^j)_{\cdot}$ are uniformly integrable.
Under the assumptions of exponential moments of any order, uniqueness has been proved first by Briand & Hu [8] and Mocha & Westray [38]. □

5.2 Existence result for BSDEs in the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$

We are now interested in quadratic BSDEs satisfying the general structure condition $|\,g(.,t,y,z)| \leqslant \kappa(t,y,z) \equiv |l_t| + c_t|y| + \frac{1}{2}|z|^2$, $d\mathbb{P} \otimes dK_{\cdot}$a.s., and are looking for solution in the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ only. As before, the method relies on a regularization of the quadratic coefficient it-self through inf-convolution as to transform it into a coefficient with *both* linear and quadratic growth. This double structure of the transformed coefficient leads to results both in terms of existence and estimation. The previous stability Theorem 4.5 can then be applied to obtain the existence of a solution, after having proved that the approximate solutions are also $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-semimartingales.

Regularization of the coefficient through inf-convolution The proof of this fundamental result is based on the following lemma involving classical regularization by inf-convolution techniques introduced by Lepeltier & San Martin [32] in a BSDEs framework. Let us first observe that the appropriate regularization when dealing with $\underline{q}(z) = -\frac{1}{2}|z|^2$ is a sup-convolution since $\underline{q}(z)$ is concave. To overcome this difficulty, we proceed in two steps, by first assuming that g is bounded from below by some basic function with both a linear and quadratic growth $\underline{\kappa}_p$, where $-\underline{\kappa}_p(t,y,z) = \kappa_p(t,y,z) \equiv l_t + c_t|y| + q_p(z)$ with $q_p(z) = \frac{1}{2}|z|^2\mathbf{1}_{\{|z|\leqslant p\}} + (p|z| - \frac{1}{2}p^2)\mathbf{1}_{\{|z|>p\}})$. When $p = 1$, $\kappa_1(t,y,z) \equiv \kappa(t,y,z) = l_t + c_t|y| + q(z)$ with $q(z) = \frac{1}{2}|z|^2$.
Lemma 5.3. *Let $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function with linear growth in y, and quadratic growth in z, bounded from below by some function $\underline{\kappa}_p(t,y,z) = -(l_t + c_t|y| + q_p(z))$ and from above by $\kappa(t,y,z)$:*

$$\underline{\kappa}_p(t,y,z) \leqslant g(t,y,z) \leqslant \kappa(t,y,z) \tag{26}$$

where the processes $c_{\cdot}$ and $l_{\cdot}$ are bounded by some universal constant $\bar{C}$.
The regularizing functions are the convex functions with linear growth $b_n(u,w) \equiv$

$n|u|+n|w|$. *The sequences* $\underline{\kappa}_{n,p}$, κ_n *and* g_n *are defined respectively as the inf-convolution of the functions* $\underline{\kappa}_p$, κ *and* g *with the function* b_n,

$$\begin{array}{ll} \underline{\kappa}_{n,p}(t,y,z) & = \underline{\kappa}_p \,\square\, b_n(t,y,z), \qquad \kappa_n(t,y,z) = \kappa \,\square\, b_n(t,y,z) \\ g_n(t,y,z) & = g \,\square\, b_n(t,y,z) = \inf_{u,w} \big(g(t,u,w) + n|y-u| + n|z-w|\big) \end{array}$$

have the following properties, for $n \geqslant \sup(\bar{C}, p)$:
(i) $\kappa_n(t,y,z) = l_t + c_t|y| + q_n(z) \leqslant l_t + c_t|y| + \frac{1}{2}|z|^2$, $\underline{\kappa}_{n,p}(t,y,z) = \underline{\kappa}_p(t,y,z)$,
(ii) $|g_n(t,y,z)| \leqslant l_t + c_t|y| + \sup(q_p(z),\, q_n(z)) = \kappa_n(t,y,z) \leqslant l_t + c_t|y| + \frac{1}{2}|z|^2$;
(iii) *the sequences* g_n *and* κ_n *are increasing;*
(iv) *the Lipschitz constant of functions* g_n *is* n;
(v) *if* $(y_n, z_n) \to (y,z)$, *then* $g_n(t, y_n, z_n) \to g(t,y,z)$.

In this lemma, the various functions are regularized through the Lipschitzian regularization, whist the function κ_n is the Moreau-Yoshida regularization of b_n (see Hiriart-Urruty & Lemaréchal [26] (Chapter E) for more details).

Existence result The important point now is to prove that the solutions to the BSDEs(g_n, ξ_T) which are Lipschitz with linear growth, are in the class $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ when $\mathbb{E}[\exp(\bar{X}_T^{\Lambda,C}(|\eta_T|))] < +\infty$.

Lemma 5.4. *Let* g *and* g_n, κ *and* κ_n *as in Lemma 5.3. The coefficients* g_n *and* κ_n *are standard uniformly Lipschitz coefficients. For any* $|\xi_T| \leqslant |\eta_T|$, *let* $(Y_.^n, Z_.^n, M_.^{n,\perp})$ *and* $(U_.^n, V_.^n, W_.^{n,\perp})$ *be the unique solution of the BSDE*(g_n, ξ_T) *and BSDE*$(\kappa_n, |\eta_T|)$ *in the appropriate space.*
(i) *The sequences* $(Y_.^n)$ *and* $(U_.^n)$ *are increasing, and satisfy the entropic inequality,* $|Y_.^n| \leqslant U_.^n \leqslant \rho_.(e^{C_{.,T}}|\eta_T| + \int_.^T e^{C_{.,t}} d\Lambda_t)$, *a.s.*
Both sequences $(Y_.^n)$ *and* $(U_.^n)$ *are* $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$*-quadratic semimartingales.*
(ii) *The sequence* $(Y_.^n, Z_.^n, M_.^{n,\perp})$ *converges uniformly in probability to a minimal solution* $(Y_., Z_., M_.^\perp)$ *of the BSDE*(g, ξ_T).

Proof. The proof relies on classical properties of BSDEs solutions associated with standard coefficients (with linear growth), in a $\mathbb{H}^2$-space. In particular, existence, uniqueness and comparison hold true in this case, that implies (i) .
(i) First assume that $\bar{X}_T^{\Lambda,C}$ is bounded. The solutions U^n are bounded and the entropic inequality is valid. Since these inequalities are stable when taking increasing limit with respect to Λ, C, η, the same inequalities hold still true under the assumption $\bar{X}_T^{\Lambda,C}(\eta_T) \in \mathbb{L}^1_{\text{exp}}$. Then, by construction, $(Y_.^n)$ and

$(U^n_\cdot)$ are $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$- quadratic semimartingales.
(ii) Finally, using Theorem 4.5, we obtain the convergence of this sequence to a solution of the BSDE(g, ξ_T) in the space $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$. $\square$

It remains to overcome the assumption made on the coefficient of a linear quadratic growth lower bound. Given a coefficient g with decomposition $g = g^+ - g^-$, where both positive functions g^+ and g^- have the same quadratic structure. Let $g_p \equiv g^+ - g^- \square b_p$. Then g_p satisfies Condition (26), and the BSDE(g_p, ξ_T) admits a minimal solution; the sequence of solutions Y^p is decreasing, and belongs to the space $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$. Once again, we use the stability theorem to conclude that the sequence Y^p converges to a solution of the BSDE(g, ξ_T). We summarize the general form of our results in the following theorem.

Theorem 5.5. *Let us consider a general BSDE(g, ξ_T), where ξ_T be a $\mathcal{F}_T$-random variable such that* $\mathbb{E}[\exp(e^{C_T}|\delta\xi_T| + \int_0^T e^{C_s} d\Lambda_s)] < +\infty$.
The coefficient $g(t, y, z)$ is satisfying the quadratic structure condition (4), $|\, g(., t, y, z)| \leqslant \frac{1}{\delta}|l_t| + c_t|y| + \frac{\delta}{2}|z|^2$.
Then, there exists at least a solution $(Y, Z, M^\perp)$ in $\mathcal{S}_Q(|\xi_T|, \Lambda, C, \delta)$ of the BSDE(g, ξ_T).

Remark 5. When both $\Lambda, C \equiv 0$, as in the framework of cash additive risk measures, the theorem simply states: if $|\, g(., t, y, z)| \leqslant \frac{\delta}{2}|z|^2$, and $\mathbb{E}[\exp(\delta|\xi_T|)] < +\infty$, their exists at least a solution in the class $(\mathcal{D}_{\exp})$.

Comment on the uniqueness of the solution The question of the uniqueness of the solution to a general quadratic BSDE is not trivial. In the standard framework where the terminal condition is bounded, Kobylanski [30] obtains the uniqueness of the solution under some Lipschitz style assumptions. Recently, Tevzadze [45] gives a direct proof of uniqueness still in the bounded case. In the case of an unbounded terminal condition, Briand & Hu [8] work under the additional assumption that the coefficient g is convex with respect to the variable z. This allows them to derive the comparison theorem, which is needed to obtain the uniqueness. Their methodology can be adapted and generalized to our framework without any particular difficulty. In a very recent paper [38], Mocha & Westray have considered general quadratic BSDEs under some stronger assumptions of exponential moment of order $p > 1$ and boundedness of the increasing processes. They obtain some interesting results for the uniqueness of the solution. The convex case

has been also studied in Delbaen, Hu & Richou [13] under weaker assumptions.

In this paper, we study the stability and convergence of some general quadratic semimartingales. The general stability result (Theorem 4.5), including the strong convergence of the martingale parts in various spaces ranging from $\mathbb{H}^1$ to BMO, is derived under some mild integrability condition on the exponential of the terminal value of the semimartingale. This strong convergence result is then used to prove the existence of solutions of general quadratic BSDEs under minimal exponential integrability assumptions, relying on a regularization in both linear-quadratic growth of the quadratic coefficient itself. On the contrary to most of the existing literature, it does not involve the seminal result of Kobylanski [30] on bounded solutions. As previously mentioned, this approach has also other potential applications such as numerical simulations of quadratic BSDEs, study in terms of risk measures and dual representation, solving of associated HJB-type equations... The various results obtained in the paper can be extended to jump processes.

Acknowledgements

Both authors would like to thank Mingyu Xu and Anis Matoussi for their helpful comments and discussions at various stages of the paper. The paper has also greatly benefited from a careful reading and helpful suggestions from two anonymous referees. Finally, the authors would like to give some special thanks to Arthur who waited patiently before arriving.

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